INTEGRATED SEMIGROUPS

Dissertation
der Mathematischen Fakultät
der Eberhard-Karls-Universität
zu Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften

vorgelegt von

Hermann Kellermann

aus Bad Wurzach

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Preface

In this paper we shall be concerned with linear Cauchy problems of first and second order.

Abstract Cauchy problems

$$u'(t) = Au(t)$$
$$u(0) = x$$

for Banach space valued functions were treated systematically first in Hille's book on functional analysis and semigroups.

Since then there were developments in various directions:

1) The techniques from operator theory were also extended to other sorts of Cauchy problems, e.g.,

a) time-dependent Cauchy problems

$$u'(t) = A(t)u(t)$$
$$u(0) = x ,$$

b) Cauchy problems of second order

$$u''(t) = A_1u'(t) + A_0u(t)$$
$$u(0) = x$$
$$u'(0) = y.$$

2) The original notion of wellposedness splitted into a huge amount of variants.

3) Many efforts were invested in applying these abstract results to partial differential equations.

This paper tries to contribute in a modest form to all three areas. To that purpose we use a concept that was

brought up about two years ago by W. Arendt.

His idea may be described as follows: Let $(e^{tA})_{t\geq 0}$ be a C_o-semigroup. Then $S_t := \int_o^t e^{rA}\, dr$ defines a family $(S_t)_{t\geq 0}$ of bounded operators having the following three properties:

(a) $S_o = 0$.

(b) $t \to S_t$ is strongly continuous.

(c) $S_s S_t = \int_o^s (S_{r+t} - S_r)\, dr$ for $s,t \geq 0$.

Now all families $(S_t)_{t\geq 0}$ of operators satisfying (a) - (c) were called "integrated semigroup".

W. Arendt could prove ([5],theorem 4.1) that any densely-defined operator $A:D(A) \to E$ on a Banach lattice E with positive resolvent $(\lambda - A)^{-1}$ for sufficiently large λ is connected with a positive and increasing integrated semigroup $(S_t)_{t\geq 0}$ by a Laplace integral

$$(\lambda - A)^{-1} = \lambda \int_o^\infty e^{-\lambda t} S_t\, dt.$$

This theorem is important in two aspects:

1) It shows that integrated semigroups yield generalized (or weak) positive solutions for certain Cauchy problems that do not lead directly to C_o-semigroups.

2) It indicates how to extend the definition of "C_o-semigroup generator" for the larger class of integrated semigroups:

An operator $A:D(A) \to E$ <u>generates</u> the integrated semigroup $(S_t)_{t\geq 0}$ if $(\lambda - A)^{-1} = \lambda \int_o^\infty e^{-\lambda t} S_t\, dt$ for all large λ.

The present paper has two goals:

In part I we try to make the structure of the integrated semigroup theory more transparent. In particular the following questions are studied:

a) Does the Laplace integral $\int_0^\infty e^{-\lambda t} S_t \, dt$ exist for all integrated semigroups? (section 3)

b) The generator of a C_0-semigroup (cosine function) is usually defined as the first (second) derivative in $t=0$. Is it possible to obtain a generator of this type also for integrated semigroups? (section 2)

c) What perturbation theorems hold for integrated semigroups? (section 6)

d) The starting point for this theory were integrated semigroups of the sort $(\int_0^t e^{rA} \, dr)_{t \geq 0}$ where $(e^{tA})_{t \geq 0}$ is a C_0-semigroup. How is an arbitrary integrated semigroup linked with a C_0-semigroup? (sections 2 and 7)

d) How are integrated semigroups linked with abstract Cauchy problems? (section 8)

In part II we try to show that integrated semigroups are of some relevance and present numerous examples.

In section 1 of part II we study differential operators of higher order and show that under very weak conditions they generate integrated semigroups.

In section 2 we shall investigate how the complete Cauchy problem of second order

$$u''(t) = A_0 u(t) + A_1 u'(t)$$
$$u(0) = x$$
$$u'(0) = y$$

can be treated with operator theoretical techniques.
Section 3 will be dedicated to some recent results about perturbations of boundary conditions ([14]). This section shows that the framework of integrated semigroups offers a technique for dealing with boundary conditions.
Section 4 shows that also Cauchy problems with inhomogeneous boundary conditions are covered by the integrated semigroup theory.

Integrated semigroups are not the only concept that extends the C_0-semigroup theory. Another one (brought up in 1960 by J.L. Lions) are the "distribution semigroups" ([16]). Since both concepts are closely related we shall, from time to time, look at their connection from different aspects.

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PART I. STRUCTURE THEORY

Section 1. Definition

Definition 1.1:

Let E be a Banach space. An integrated semigroup is a family $(S_t)_{t\geq 0}$ of linear bounded operators S_t on E with the following properties:

(a) $S_0 = 0$.

(b) $t \to S_t$ is strongly continuous.

(c) $S_s S_t = \int_0^s (S_{r+t} - S_r)\,dr$ for $s,t \geq 0$.

Remarks, Examples:

1) The background of the term "integrated semigroup" is this: Let $(e^{tA})_{t\geq 0}$ a C_0-semigroup. The family $(\int_0^t e^{sA}\,ds)_{t\geq 0}$ fulfills properties (a) - (c), i.e. integrated C_0-semigroups are integrated semigroups.

2) Let P be a bounded projection on a Banach space E. $(S_t)_{t\geq 0}$, where $S_t := tP$ is an integrated semigroup. If $P \neq I$ then $(S_t)_{t\geq 0}$ is not an integrated C_0-semigroup. Moreover $(S_t)_{t\geq 0}$ has a property that plays a disturbing role throughout the whole theory: $(S_t)_{t\geq 0}$ vanishes on a nontrivial subspace. Later we will call that property "degeneracy" and exclude these integrated semigroups from further considerations.

3) Let $(T_t)_{t\geq 0}$ be a C_o-semigroup on a Banach space E. $(S_t)_{t\geq 0} := (\int_o^t T_s^* \, ds)_{t\geq 0}$ (where T_s^* is the adjoint of T_s) is an integrated semigroup on E'. In general $(S_t)_{t\geq 0}$ is not an integrated C_o-semigroup.

4) Let $(\mathrm{Cos}(t))_{t\in\mathbb{R}}$ be a cosine operator function on a Banach space E (see e.g. [12]). If $\mathrm{Sin}(t) := \int_o^t \mathrm{Cos}(s)ds$ then by

$$S_t = \begin{pmatrix} \mathrm{Sin}(t) & \int_o^t \mathrm{Sin}(s)\, ds \\ \mathrm{Cos}(t) - I & \mathrm{Sin}(t) \end{pmatrix}$$

an integrated semigroup $(S_t)_{t\geq 0}$ is given on $E \times E$.
This and more about Cauchy problems of second order will be proved in part II, section 2.

5) A short calculation shows that the functional equation 1.1(c) can also be written as

$$S_s S_t = \int_o^{s+t} S_r \, dr - \int_o^s S_r \, dr - \int_o^t S_r \, dr \quad \text{for } s,t \geq 0.$$

A consequence of this: $S_s S_t = S_t S_s$ for $s,t \geq 0$.

6) In the mean time definition 1.1 has been generalized again: If we iterate the heuristical integration procedure that we described in the preface we arrive at "q-times integrated semigroups":

A family $(S_t)_{t\geq 0}$ of bounded linear operators is a <u>q-times integrated semigroup</u> if it fulfills 1.1(a), (b) and

$$(c^q) \quad \begin{aligned} S_s S_t &= \frac{1}{(q-1)!} \int_o^s \left((s-r)^{q-1} S_{r+t} - (s+t-r)^{q-1} S_r\right) dr \\ &= \frac{1}{(q-1)!} \int_o^{s+t} (s+t-r)^{q-1} S_r \, dr \\ &\quad - \frac{1}{(q-1)!} \int_o^s (s+t-r)^{q-1} S_r \, dr \\ &\quad - \frac{1}{(q-1)!} \int_o^t (s+t-r)^{q-1} S_r \, dr \quad \text{for } s,t \geq 0. \end{aligned}$$

For more about q-times integrated semigroups see [21].

7) In the following we shall repeatedly differentiate operator functions $t \to F_t$. This differentiation will always be done with respect to the strong operator topology. For shortness we will not mention this every time. So, if an operator function is called "(continuously) differentiable" this means "strongly (continuously) differentiable".

Section 2. Differentiation spaces and generator for integrated semigroups

In this and the following section we will try to answer the question:

What is a meaningful definition of a generator?

The answer is fairly obvious in the case of integrated C_o-semigroups, namely: The generator of the corresponding C_o-semigroup that can be obtained in two ways:

1) By differentiating twice the function

$$t \to S_t = \int_o^t e^{sA}\, ds \quad \text{in } t = 0 .$$

2) By calculating the resolvent as a Laplace integral

$$R_\lambda = \lambda \int_o^\infty e^{-\lambda t} S_t\, dt \quad \text{and then by} \quad A := \lambda - R_\lambda^{-1}.$$

Going over to the general case with either approach one experiences a surprise:

1) and 2) yield two different answers. We shall follow 1) in this section and 2) in section 3.

Let, from now on, E always be a Banach space and $(S_t)_{t \geq 0}$ an integrated semigroup on E.

Definition 2.1: Let $n \geq 0$.

$C^n := \{ x \in E : t \to S_t x$ is n-times continuously differentiable$\}$

is called the n-th differentiation space.

On C^n the n-th derivative $S_t^{(n)}$ can be defined (as an operator with values in E).

Proposition 2.2:

The following holds:

(i) $S_t C^n \subset C^{n+1}$ for $n \geq 0$,

(ii) $S_t' C^n \subset C^n$ for $n \geq 1$,

(iii) $S_t'' C^n \subset C^{n-1}$ for $n \geq 2$.

Beweis:

(i): Let $x \in C^n$.

We have to show: $s \to S_s S_t x$ is a C^{n+1}-function.

According to the functional equation 1.1(c) we have

$$S_s S_t x = \int_0^s (S_{r+t}x - S_r x)\,dr$$
$$= \int_0^{s+t} S_r x\,dr - \int_0^s S_r x\,dr - \int_0^t S_r x\,dr .$$

The right hand side is (n+1)- times continuously differentiable in s. Therefore (i) holds.

(ii): Let $x \in C^n$, $n \geq 1$. From the functional equation we obtain $S_s S_t' x = S_{s+t}x - S_t x$ and (ii) by the same argument.

(iii): Let $x \in C^n$, $n \geq 2$. Differentiating the previous equation we obtain $S_s S_t'' x = S_{s+t}' x - S_t' x$ and again the assertion.

Proposition 2.3 (Properties of $S'_t : C^1 \to C^1$):

(i) $S'_s S'_t = S'_{s+t}$ for $s,t \geq 0$.

(ii) S'_o is a projection.

(iii) $S'_o S'_t = S'_t$.

(iv) $S_s S'_t = S'_t S_s$ for $s,t \geq 0$.

Proof: All properties can be derived by differentiating the functional equation.

Remark 2.4:

2.3(i) and (ii) raise the question:

Is, in case $S'_o = I_{|C^1}$, $(S'_t)_{t\geq 0}$ a C_o-semigroup on C^1?

Or, to state the question in a more precise form:

If $S'_o = I_{|C^1}$ is then a) C^1 $\| \ \|_E$-closed and b) $(S'_t)_{t\geq 0}$ a C_o-semigroup on $(C^1, \| \ \|_E)$?

This is "almost never" true if

$$S_t = \begin{pmatrix} \mathrm{Sin}(t) & \int_o^t \mathrm{Sin}(s)\, ds \\ \mathrm{Cos}(t) - I & \mathrm{Sin}(t) \end{pmatrix}$$

(compare section 1, remark 4). A specific example is the cosine operator function on $C_o(\mathbb{R})$, given by

$$\mathrm{Cos}(t)f(x) = \tfrac{1}{2}(f(x + t) + f(x - t))$$

(the cosine-function connected with the wave equation). Here $(S_t)_{t\geq 0}$ is not continuously differentiable on the whole space $E = C_o(\mathbb{R}) \times C_o(\mathbb{R})$ but on the dense subspace $\{f \in C_o(\mathbb{R}) : f \in C^1 \text{ and } f' \in C_o(\mathbb{R})\} \times C_o(\mathbb{R})$. Therefore $\overline{C^1} = E \neq C^1$ or: C^1 is not closed (and a) does not hold).

One might weaken the question and ask:

Can, in case $S'_o = I_{|C^1}$, $(S'_t)_{t\geq 0}$ be extended to a C_o-semigroup on $(\overline{C^1}, \| \|_E)$?

The following theorem shows that this does not lead any further.

Theorem 2.5:

Let $(S_t)_{t\geq 0}$ be an integrated semigroup.

Equivalent are:

(i) C^1 is $\| \|_E$-closed.

(ii) $\| S'_t \|$ is bounded on compact t-intervals.

So, if $(S'_t)_{t\geq 0}$ can be extended to a C_o-semigroup on $(\overline{C^1}, \| \|_E)$ then by theorem 2.5 C^1 is already $\| \|_E$-closed and we wind up at the original question.

Theorem 2.5 shows on the other hand: If (i) or (ii) is fulfilled then $(S'_t)_{t\geq 0}$ is a C_o-semigroup on $(C^1, \| \|_E)$ (provided that $S'_o = I_{|C^1}$).

Proof of theorem 2.5:

(i)=>(ii): By the Banach-Steinhaus theorem we obtain: For every $t \geq 0$ S'_t is a bounded operator, being the strong limit of a sequence of bounded operators. And since $t \to S'_t$ is strongly continuous (definition of C^1!) (ii) follows from the uniform boundedness theorem.

(ii)=>(i): Let (x_n) be a sequence in C^1 with $x_n \to x$. Let $c > 0$. Then: $S_t x_n \to S_t x$ and $S'_t x_n \to g(t)$ uni-

formly for $0 \le t \le c$ and for some function $g:[0,c] \to E$ (because of (ii)).

Therefore $t \to S_t x$ is a C^1-function on $[0,c]$ for arbitrary $c > 0$. In other words: $x \in C^1$.

Before we turn to the question "Is $S_o'':C^2 \to C^1$ a reasonable generator?" we state a few properties of S_o''.

Proposition 2.6:

(i) $S_t' = S_o'' S_t + S_o'$ on C^1.

(ii) $S_t S_o'' = S_o'' S_t$ and

$S_t'' = S_o'' S_t'$ on C^2.

(iii) $S_o^{(n)} = (S_o'')^{n-1}$

$S_t^{(n)} = S_o^{(n)} S_t'$ on C^n, $n \ge 2$.

Proof: All equations can be obtained by differentiating the functional equation 1.1(c) sufficiently often.

A reasonable generator should determine the integrated semigroup.

Does S_o'' determine the integrated semigroup?

No.

Example 2.7:

Take $E = \mathbb{R}$, $S_t = t$ and $T_t = 0$. Here $S_o'' = T_o'' = 0$.

In this case we have $S_o' = I \neq 0 = T_o'$.

This suggests the question:

Does the pair (S''_o, S'_o) determine the integrated semigroup?

No.

Example 2.8:

Choose an unbounded generator $A:D(A) \to E$ of a C_o-semigroup and an injective linear mapping $B:\mathbb{C} \to E$, such that $B(\mathbb{C}) \cap D(A) = \{0\}$.

By

$$S_t^B = \begin{pmatrix} \int_o^t e^{sA}\,ds & (e^{tA} - I)B \\ 0 & 0 \end{pmatrix}$$

an integrated semigroup is given on $E \times \mathbb{C}$.

$$S'_o = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on } C^1 = E \times \{0\}, \qquad S''_o = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on } C^2 = D(A) \times \{0\}.$$

This holds for any such B. But there are enough B's to get two different integrated semigroups $(S_t^B)_{t \geq 0}$.

At least the following holds:

Theorem 2.9:

S_o' and S_o'' determine the integrated semigroup $(S_t)_{t\geq 0}$ on $\overline{C^1}$.

More precisely: Let $(S_t)_{t\geq 0}$ be an integrated semigroup with C^1 and C^2 as defined in 2.1.

Every integrated semigroup $(T_t)_{t\geq 0}$ that is continuously differentiable on C^1 with $T_o' = S_o'$ and twice continuously differentiable on C^2 with $T_o'' = S_o''$ coincides with $(S_t)_{t\geq 0}$ on $\overline{C^1}$.

Proof:

Let $(T_t)_{t\geq 0}$ be such an integrated semigroup and $x \in C^1$. The function $s \to T_{t-s}S_s\, x$ is continuously differentiable and

$$\begin{aligned}\frac{d}{ds}(T_{t-s}S_s\, x) &= -T'_{t-s}\, S_s\, x + T_{t-s}\, S'_s\, x \\ &= -T_{t-s}T''_o S_s x - T'_o S_s x + T_{t-s}(S''_o S_s x + S'_o x) \\ &\qquad \text{(prop. 2.6(i) und (ii))} \\ &= T_{t-s}x - S_s x \\ &\qquad \text{(by the assumptions and prop. 2.3.(iii))}.\end{aligned}$$

Integration from $s = 0$ to $s = t$ yields

$$\int_o^t \frac{d}{ds}(T_{t-s}S_s\, x)\, ds = \int_o^t T_{t-s}x\, ds - \int_o^t S_s x\, ds.$$

Applying the fundamental theorem of calculus on the left hand side and using $S_o = T_o = 0$ we obtain

$$\int_o^t T_s x\, ds = \int_o^t S_s x\, ds\ .$$

Differentiation yields

$$T_t x = S_t x \quad \text{for } x \in C^1\ .$$

The assertion follows on $\overline{C^1}$ from the boundedness of S_t and T_t.

Conclusion 2.10:

Example 2.7 showed that S_o'' alone does not determine the integrated semigroup. And even if S_o' is known (in fact, in most applications $S_o' = I_{|C^1}$) S_o'' is only of limited use: We need additionally that C^1 is dense in E. In part II we shall see that there is an important class of integrated semigroups with non-dense C^1.

On the other hand density relations are an important tool in operator theory. Therefore we ask:

What density relations hold for the differentiation spaces C^n?

To answer this we equip C^n with the norm $\| \ \|_{n-1}$, defined by

$$\begin{aligned} \|x\|_{n-1} &:= \|x\| + \|S_o'x\| + \|S_o''x\| + \ldots + \|S_o^{(n)}x\| \\ &= \|x\| + \|S_o'x\| + \|S_o''x\| + \|(S_o'')^2x\| + \ldots \\ &\qquad \ldots + \|(S_o'')^{n-1}x\| . \end{aligned}$$

(by prop. 2.6(iii))

In general $(C^n, \| \ \|_{n-1})$ is not a Banach space (see the example in remark 2.4).

From the second line in the definition we see:

If $S_o' = I_{|C^1}$ the norm $\| \ \|_{n-1}$ is equivalent to the (n-1)-th graphnorm of S_o''. This is the reason for the index n-1.

Definition 2.11:

Let $(S_t)_{t\geq 0}$ be an integrated semigroup.

The set $N := \{ x \in E : S_t x = 0 \text{ für alle } t \geq 0 \}$

is called the degeneration space of $(S_t)_{t\geq 0}$.

$(S_t)_{t\geq 0}$ is called non-degenerate, if $N = \{0\}$,

otherwise degenerate.

Proposition 2.12:

(i) $N \subset C^n$ for all $n \in \mathbb{N}$.

(ii) $N = \ker S_0'$.

In particular: $(S_t)_{t\geq 0}$ is non-degenerate if only if $S_0' = I_{|C^1}$.

Proof: (i) is obvious, and also $\subset$ in (ii).

$\ker S_0' \subset N$: Let $x \in C^1$ with $S_0'x = 0$. By 2.3(iii) $S_t x = S_t S_0' x = 0$ für $t \geq 0$, therefore $x \in N$.

The equivalency follows from S_0' being a projection (prop. 2.3.(ii)).

Theorem 2.13:

Let $n \geq 1$.

C^{n+1} is dense in C^n (with respect to the norm $\| \|_{n-1}$).

Proof:

We prove first: $S_0'C^{n+1}$ is dense in $S_0'C^n$.

Let $x \in C^n$. Then $(S_h/h)S_0'x = S_0'(S_h/h)x \in S_0'C^{n+1}$ (prop. 2.2.(i)).

We have to show: $\|(S_h/h)S_o'x - S_o'x\|_{n-1} \to 0$ for $h \to 0$.

Or: $S_o^{(k)}(S_h/h)x = (S_h/h)S_o^{(k)}x \to S_o^{(k)}x$ for $1 \le k \le n-1$.

For $k = 1$ this follows from prop. 2.2(ii) and the definition of C^1.

For $2 \le k \le n-1$ $S_o^{(k)}x = (S_o'')^{k-1}x \in C^1$ (prop. 2.6(iii) and prop. 2.2(iii)). Therefore the convergence follows in this case, too.

Since S_o' is a projection we have for all $n \in \mathbb{N}$

$$\begin{aligned} C^n &= \ker(S_o'|C^n) \oplus S_o'C^n \\ &= \ker S_o' \cap C^n \oplus S_o'C^n \\ &= N \oplus S_o'C^n \quad \text{(by prop. 2.12(ii))}. \end{aligned}$$

This completes the proof.

We dedicate the rest of this section to the question:
How are q-times integrated semigroups linked with distribution semigroups?
For the definition of distribution semigroups we refer to [16].

We start with some preparatory material about q-times integrated semigroups.
The definitions of differentiation spaces and derivatives (def. 2.1) and of non-degeneracy (def. 2.11) can be extended for q-times integrated semigroups in the obvious way.
With this the following holds:

Lemma 2.14:

Let $(S_t)_{t\geq 0}$ be a non-degenerate q-times integrated semigroup.

Then

(i) $S_o^{(k)} = 0$ for $k \leq q - 1$,

(ii) $S_o^{(q)} = I_{|C^q}$.

Proof:

Using the functional equation (c^q) in remark 1.2(6) one shows by induction

$$S_t S_s^{(k)} = \frac{1}{(q-1-k)!} \int_o^s ((s - r)^{q-1-k} S_{r+t} - (s + t - r)^{q-1-k} S_r)\, dr$$

$$(*) \qquad - \sum_{j=o}^{k-1} \frac{1}{(j+q-k)!} t^{j+q-k} S_s^{(j)}$$

for $k \leq q - 1$ and $s,t \geq 0$.

For $k = 1$ we obtain from this $S_t S_o' = 0$ for $t \geq 0$ (if $q > 1$). By non-degeneracy $S_o' = 0$.

Repeating this argument for $2 \leq k \leq q - 1$ we deduce (i).

Differentiating (*) we obtain

$$S_t S_s^{(q)} = S_{s+t} - \sum_{j=o}^{q-1} \frac{1}{j!} t^j S_s^{(j)} \quad \text{for } s,t \geq 0.$$

Therefore by (i) $S_t S_o^{(q)} = S_t$ for $t \geq 0$ or $S_t(S_o^{(q)} - I_{|C^q}) = 0$. Non-degeneracy implies (ii).

The linkage between q-times semigroups and distribution semigroups is partially described by

Theorem 2.15:

Let $S := (S_t)_{t \geq 0}$ be a non-degenerate q-times integrated semigroup with the property that C^q is dense in E.

Then

$$G(\phi) := \int_0^\infty (-1)^q \phi^{(q)}(t)\, S_t\, dt$$

defines a distribution semigroup on E.

In other words: The q-th distributional derivative $D^q S$ is a distribution semigroup.

Proof:

We just show that the set

$F := \{ G(\phi)x : x \in E, \phi \in \mathcal{D}(0,\infty) \}$ is dense in E (property 1.1(iv) in [16]). All other properties can be verified easily.

Since by assumption C^q is dense in E we only have to show that every $x \in C^q$ can be approximated by elements in F. For $x \in C^q$ and $\phi \in \mathcal{D}(0,\infty)$ we have

$$G(\phi)x = \int_0^\infty \phi(t)\, S_t^{(q)} x\, dt.$$

Choose a sequence (ϕ_n) in $\mathcal{D}(0,\infty)$ such that $\phi_n \geq 0$, supp $\phi_n \subset (0,\frac{1}{n})$ and $\int_0^\infty \phi_n(t)\, dt = 1$. Then

$$\|G(\phi_n)x - x\| = \|\int_0^\infty \phi_n(t)(S_t^{(q)} x - x)\, dt\|$$
$$\leq \sup_{t \leq 1/n} \|S_t^{(q)} x - x\|.$$

By lemma 2.14(ii) $\|G(\phi_n)x - x\| \to 0$ as $n \to \infty$.

The following example shows that not all distribution semigroups are the q-th distributional derivative of a q-times integrated semigroup.

Example 2.16:

Let $E = L^2(\mathbb{R}_+)$ and $g:\mathbb{R}_+ \to \mathbb{C}$, $g(x) := x + ie^x$.

$G(\phi) := \int_0^\infty (-1)^m \phi^{(m)}(t) \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} e^{sg} \, ds \, dt$

for $\operatorname{supp} \phi \subset (-\infty, m]$

defines a distribution semigroup on E. (The right hand side is understood as a multiplier.)

G is not the q-th distributional derivative of a q-times integrated semigroup.

Proof:

G is welldefined, i.e. independent of the choice of m which can be seen by an integration by parts.

We do not show all properties of distribution semigroups but restrict ourselves to property 1.1(i) in [16]:

$G \in \mathcal{D}'_+(L(E))$.

For $\phi \in \mathcal{D}_-$ with $\operatorname{supp} \phi \subset (-\infty, m]$ we have

$$G(\phi) = \int_0^\infty (-1)^m \phi^{(m)}(t) \left(e^{tg}/g^m - \Sigma_{j=0}^{m-1} \frac{1}{j!} t^j \left(\frac{1}{g}\right)^{m-j}\right) dt.$$

Using the definition of g we obtain the estimate

$$\left|e^{tg(x)}/g(x)^m - \Sigma_{j=0}^{m-1} \frac{1}{j!} t^j \left(\frac{1}{g}\right)^{m-j}\right| \leq e^{(t-m)x} + \Sigma_{j=0}^{m-1} \frac{1}{j!} t^j \quad \text{for } x \geq 0.$$

Therefore $\|G(\phi)\|_{L(E)} \leq m \|\phi^{(m)}\|_\infty (1 + \Sigma_{j=0}^{m-1} \frac{1}{j!} m^j)$ if $\operatorname{supp} \phi \subset (-\infty, m]$.

This shows $G(\phi) \in L(E)$ and continuity of the mapping $\phi \to G(\phi)$.

Since all other properties hold, too, (as can be seen in straight forward calculations) G is a distribution semigroup.

Suppose now that there is a q-times integrated semigroup $(S_t)_{t\geq 0}$ such that $G(\phi) = \int_0^\infty (-1)^q \phi^{(q)}(t)\, S_t\, dt$ for all $\phi \in \mathcal{D}_-$.

Integration by parts yields

$$G(\phi) = \int_0^\infty (-1)^{q+1}\, \phi^{(q+1)}(t) \int_0^t S_s\, ds\, dt.$$

On the other hand

$$G(\phi) = \int_0^\infty (-1)^{q+1}\, \phi^{(q+1)}(t)\, \frac{1}{q!} \int_0^t (t-s)^q\, e^{sg}\, ds\, dt$$

for $\operatorname{supp} \phi \subset (-\infty, q+1]$.

This shows

$$\int_0^t S_s\, ds = \frac{1}{q!} \int_0^t (t-s)^q\, e^{sg}\, ds \quad \text{for } 0 \leq t \leq q+1.$$

Therefore

$$\begin{aligned} S_t &= \frac{1}{(q-1)!} \int_0^t (t-s)^{q-1}\, e^{sg}\, ds \\ &= e^{tg}/g^q - \sum_{j=0}^{q-1} \frac{1}{j!}\, t^j \left(\frac{1}{g}\right)^{q-j} \quad \text{for } 0 \leq t \leq q+1. \end{aligned}$$

After a look at the left and the right hand side we conclude that e^{tg}/g^q is a bounded multiplier on E for $0 \leq t \leq q+1$.

But $|e^{tg(x)}/g(x)^q| \geq e^{tx}/|x + ie^x|^q$

$$\geq e^{tx}/e^{qx} = e^{(t-q)x}.$$

By choosing $t > q$ we obtain an unbounded multiplier and a contradiction.

So q-times integrated semigroups do not cover all distribution semigroups. But we shall see in section 4 that they cover an important class of them (remark 4.7).

Section 3. The Laplace integral

In this section we shall try to define the generator by $A := \lambda - R_\lambda^{-1}$ where

$$R_\lambda = \lambda \int_0^\infty e^{-\lambda t} S_t \, dt .$$

The first question is: Does this integral always exist as a strong or weak integral?

No.

Example 3.1:

$E = \ell^2$, $S_t : (x_n) \to \left(\int_0^t \exp(a_n s) \, ds \, x_n \right)$, where

$a_n := n + 2^{n^2} \pi i$.

$(S_t)_{t \geq 0}$ is an integrated semigroup:

1) $S_t \in L(E)$: $\left| \int_0^t \exp(a_n s) \, ds \right| = |\exp(a_n t) - 1| / |a_n|$

$\leq (e^{nt} + 1) / e^{n^2 \log 2} \leq 2e^{nt} / e^{n^2 \log 2}$

$= 2e^{nt - n^2 \log 2} \leq 2e^{t^2/4 \log 2}$

2) $t \to S_t$ is strongly continuous:

Obviously $t \to S_t e_n$, where $e_n = (\delta_{in})$, is continuous. Since $\{e_n : n \in \mathbb{N}\}$ is total in ℓ^2 and $(S_t)_{t \geq 0}$ is uniformly bounded on compact t-intervals (see 1)) the strong continuity of $(S_t)_{t \geq 0}$ follows.

3) The functional equation is clearly fulfilled and $S_o = 0$.

Therefore $(S_t)_{t \geq 0}$ is an integrated semigroup.

But: For all $\lambda \in \mathbb{C}$ the Laplace integral does not even converge as a weak integral.

For given $\lambda \in \mathbb{C}$ $\langle \int_0^\alpha e^{-\lambda t} S_t \, dt \, x,y \rangle$ diverges (as $\alpha \to \infty$) for $x = (1/\overline{a_n})$ and $y = (1/(a_n - \lambda))$:

$$\langle \int_0^\alpha e^{-\lambda t} S_t \, dt \, x,y \rangle$$
$$= \Sigma_{n=1}^{\infty} \int_0^\alpha e^{-\lambda t} \int_0^t \exp(a_n s) \, ds \, dt \, / \overline{a_n}(\overline{a_n - \lambda})$$

and after a short calculation

$$= \Sigma_{n=1}^{\infty} \exp((a_n - \lambda)\alpha)/(|a_n||a_n - \lambda|)^2$$
$$- \Sigma_{n=1}^{\infty} 1/(|a_n||a_n - \lambda|)^2$$
$$- \Sigma_{n=1}^{\infty} \int_0^\alpha e^{-\lambda t} \, dt/(|a_n|^2(\overline{a_n - \lambda})).$$

The last two terms converge for $\alpha \to \infty$, therefore we only consider the first term. For $\alpha \in \mathbb{N}$ we obtain by the definition of a_n

$$\Sigma_{n=1}^{\infty} \exp((a_n - \lambda)\alpha)/(|a_n||a_n - \lambda|)^2$$
$$= e^{-i(\mathrm{Im}\lambda)\alpha} \Sigma_{n=1}^{\infty} e^{(n - \mathrm{Re}\lambda)\alpha}/(|a_n||a_n - \lambda|)^2.$$

For $\alpha \to \infty$ every summand with index $n > \mathrm{Re}\lambda$ diverges to infinity therefore the series diverges also.

This example shows that

a) even on Hilbert spaces the Laplace integral does not always converge,

b) the Laplace integral does not even converge for norm continuous integrated semigroups. A more detailed estimate shows that this integrated semigroup is even continuous with respect to the operator norm.

On the other hand: We shall see that in the vast majority of applications the Laplace integral converges for all λ in a right halfplane because most integrated semigroups have the following property:

For suitable constants M and $\omega \in \mathbb{R}$

$$\|S_t\| \leq Me^{\omega t} \quad \text{for } t \geq 0 .$$

We call such integrated semigroups exponentially bounded. Then the Laplace integral $R_\lambda := \lambda\int_0^\infty e^{-\lambda t}S_t\,dt$ exists for all λ with $Re\lambda > \omega$, but - in contrast to C_0-semigroups - it is not always injective (e.g. $S_t = 0$).
From now on we shall confine ourselves to exponentially bounded integrated semigroups.

Proposition 3.2:

(i) $(R_\lambda)_{\lambda>\omega}$ is a pseudo-resolvent.

(ii) $\ker R_\lambda = N$.

Proof:

(i): see [3], theorem 3.1.

(ii): Obviously $N \subset \ker R_\lambda$.

$\ker R_\lambda \subset N$: Since $(R_\lambda)_{\lambda>\omega}$ is a pseudo-resolvent $\ker R_\lambda$ is independent of λ. Let $x \in \ker R_\lambda$. Then

$R_\lambda = \lambda\int_0^\infty e^{-\lambda t}S_t\,dt = 0$ for all $\lambda > \omega$.

The uniqueness theorem for Laplace transforms implies $S_t x = 0$ for all $t \geq 0$.

Remark 3.3:

In [3] W. Arendt proved even more than 3.2(i):

Let $(S_t)_{t\geq 0}$ be a strongly continuous family of bounded linear operators S_t with $\|S_t\| \leq Me^{\omega t}$ for $t \geq 0$ and $S_0 = 0$. Let $R_\lambda := \lambda\int_0^\infty e^{-\lambda t}S_t\,dt$ for $\lambda > \omega$.

Equivalent are

(i) $(S_t)_{t\geq 0}$ is an integrated semigroup.

(ii) $(R_\lambda)_{\lambda>\omega}$ is a pseudo-resolvent.

In section 6 this theorem will be an important tool in proving perturbation theorems.

Other properties of R_λ are contained in

Proposition 3.4:

(i) R_λ commutes with S_t and all derivatives, i.e.
$$S_t^{(n)} R_\lambda = R_\lambda S_t^{(n)} \quad \text{on } C^n.$$

(ii) $R_\lambda E \subset C^1$ and
$$N \oplus R_\lambda C^n = C^{n+1} \quad \text{for } n \geq 1 .$$

(iii) $S_o' R_\lambda = R_\lambda$.

(iv) $(\lambda - S_o'') R_\lambda = S_o'$ on C^1.

Proof:

Obviously $S_t R_\lambda = R_\lambda S_t$. Therefore (i). Moreover, using the definition of R_λ and the functional equation 1.1(c) one can show

$$S_s R_\lambda = \int_o^s e^{\lambda r} \, dr R_\lambda - \int_o^s e^{\lambda r} \lambda \int_o^r e^{-\lambda t} S_t \, dt dr - \int_o^s S_r \, dr.$$

From this we obtain $R_\lambda C^n \subset C^{n+1}$ for $n \geq 0$ and by differentiating this equation we get (iii) and (iv).

A close look at (vi) (with (i) and (iii) in mind) shows: $R_\lambda | S_o' C^n$ is the inverse of $(\lambda - S_o'') | S_o' C^{n+1}$. In particular $R_\lambda S_o' C^n = S_o' C^{n+1}$.

In the proof of theorem 2.13 we showed that

$$C^{n+1} = N \oplus S_o' C^{n+1} .$$

Hence
$$C^{n+1} = N \oplus R_\lambda S_o' C^n$$
$$= N \oplus R_\lambda C^n$$

since $R_\lambda S_o' = S_o' R_\lambda = R_\lambda$ (by (i) and (iii)).

Remark 3.5:

Let $(S_t)_{t\geq 0}$ be an integrated semigroup that is exponentially bounded (with constants M and ω) and non-degenerate.

Then we obtain from prop. 3.4:

For $\mathrm{Re}\ \lambda > \omega$

a) $R_\lambda\colon C^1 \to C^2$ is the inverse of $(\lambda - S_o'')$,

b) $C^2 \subset R_\lambda E \subset C^1$.

By prop. 3.2 $(R_\lambda)_{\mathrm{Re}\lambda>\omega}$ is a resolvent. Therefore there is an operator $A:D(A) \to E$ with $R_\lambda = (\lambda - A)^{-1}$.

With this a) und b) take the form:

a') $C^2 \subset D(A) \subset C^1$,

b') $A_{|C^2} = S_o''$.

Since $C^{n+1} = R_\lambda C^n$ (prop. 3.4(ii)) and $D(A^{n+1}) = R_\lambda D(A^n)$ we get more generally:

a") $C^{n+1} \subset D(A^n) \subset C^n$ for $n \geq 1$.

a") and b') have several consequences:

c) The n-th graph norm of A (defined by $\|x\| + \|Ax\| + \ldots + \|A^n x\|$ for $x \in D(A^n)$) is on C^{n+1} equivalent to the norm $\|\ \|_n$.

Since from now on we shall only consider integrated semigroups that are exponentially bounded and non-degenerate we simplify our notation by redefining $\|\ \|_n$:

$$\|x\|_n := \|x\| + \|Ax\| + \ldots + \|A^n x\|$$

In other words: We extend $\|\ \|_n$ to the larger space $D(A^n)$ and drop the term $\|S_o' x\| = \|x\|$.

Since $C^{n+1} = R_\lambda C^n$ implies $(\lambda - A)C^{n+1} = C^n$ and since $\lambda - A$ is an isomorphism from $(D(A^n), \|\ \|_n)$ onto

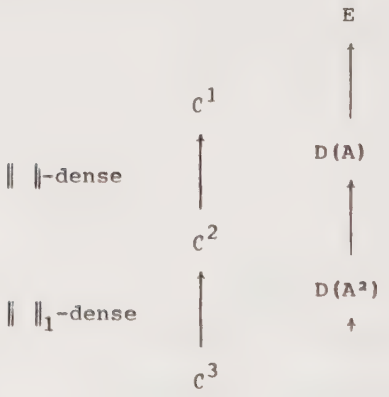
$(D(A^{n-1}), \| \|_{n-1})$ we obtain

d) $\lambda - A$ is an isomorphism from $(C^{n+1}, \| \|_n)$ onto $(C^n, \| \|_{n-1})$ for $n \geq 1$.

In theorem 2.13 we proved

e) C^{n+1} is $\| \|_{n-1}$-dense in C^n.

We can put b"), d) and e) together to the following diagram.

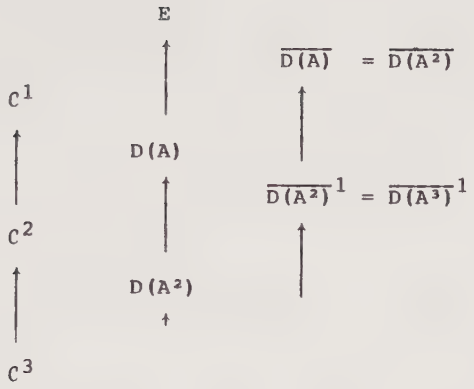


This should be read like this:

All sets in the diagram contain the sets underneath themselves ($\ldots D(A^2) \subset C^2 \subset D(A) \subset \ldots$). The arrows denote the isomorphisms $\lambda - A$.

All inclusions may be proper. In part II this will be illustrated with several examples.

If we denote the $\| \|_n$-closure of a space F by $\bar{F}^n$ (which we will also do in the future) this diagram translates to



Two inclusions will play a particular role in the future: $D(A) \subset C^1 \subset \overline{D(A)}$.

We shall come back several times to these diagrams to use them in proofs (part I) and to interpret them in terms of well-known and familiar structures (part II).

Section 4. Characterization of L-generators

The introductory question in section 2 was the question how a generator can be defined.

We had a closer look at two operators:

a) $S_o'': C^2 \to E$,

b) $A: D(A) \to E$ linked with $(S_t)_{t\geq 0}$ by the Laplace integral.

The operator S_o'' can, unlike A, always be defined. But in general it does not determine the integrated semigroup. Together with $S_o': C^1 \to E$ it characterizes $(S_t)_{t\geq 0}$ at least in the case when C^1 is dense in E.

If one restricts oneself to integrated semigroups that are exponentially bounded and non-degenerate the operator A certainly deserves the title "generator".

Although these integrated semigroups are the most important we leave this definition problem open and say instead:

A is the Laplace generator (or shorter: L-generator) of the integrated semigroup $(S_t)_{t\geq 0}$ if $(S_t)_{t\geq 0}$ is exponentially bounded and $(\lambda - A)^{-1} = \lambda \int_o^\infty e^{-\lambda t} S_t \, dt$ for all large enough λ. (Since we shall not use the operator S_o'' very extensively we do not give it a name.)

So far it was not possible to develop a "Hille-Yoshida-theory" for L-generators i.e. a theory that completely characterizes L-generators. This could be done, though,

for densely defined L-generators (see theorem 4.6). We present a few results that we shall need later.
The basis of these theorems is a vector-valued version of a theorem by Widder that W. Arendt proved for Banach spaces.

Theorem 4.1 (Arendt [4]):

Let $r:(0,\infty) \to E$ be a Banach space valued function and $c \geq 0$.

Equivalent are:

(i) There is a function $v:[0,\infty) \to E$ with the properties

a) $v(0) = 0$ and

$$\|v_t - v_s\| \leq c|t - s| \quad \text{for } 0 \leq s,t,$$

b) $r_\lambda = \lambda \int_0^\infty e^{-\lambda t} v(t)\, dt$ for all $\lambda > 0$.

(ii) r is infinitely often differentiable and for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda > 0$

$$\|(\frac{d}{d\lambda})^n r_\lambda\| \leq n!c/\lambda^{n+1} .$$

Using this theorem W. Arendt proved the following characterization theorem.

Theorem 4.2 (Arendt [4]):

Let $A:D(A) \to E$ linear, $c \geq 0$.

Equivalent are:

(i) A is L-generator of an integrated semigroup semigroup $(S_t)_{t\geq 0}$ with

$$\|S_t - S_s\| \leq c|t - s| \quad \text{for } 0 \leq s,t.$$

(ii) $(0,\infty) \subset \rho(A)$ and

for all $n \in \mathbb{N}$ and $\lambda > 0$

$$\|R_\lambda^n\| \leq c/\lambda^n .$$

A variant of this is

Theorem 4.3:

Equivalent are:

(i) $A:D(A) \to E$ is the L-generator of an integrated semigroup that is locally Lipschitz-continuous (with respect to the operator norm).

(ii) There are constants M, ω such that:

$(\omega,\infty) \subset \rho(A)$ and for all $n \in \mathbb{N}$

$$\|R_\lambda^n\| \leq M/(\lambda - \omega)^n \quad \text{for } \lambda > \omega .$$

Proof:

We prove theorem 4.3 by reducing it to theorem 4.2.

(i) => (ii):

Let $(S_t)_{t\geq 0}$ be the integrated semigroup for A. Since $(S_t)_{t\geq 0}$ is locally Lipschitz-continuous $\|S_t'\|$ is uniformly bounded on compact t-intervals. By theorem 2.5 C^1 is closed and by the remark after theorem 2.5 $(S_t')_{t\geq 0}$ is a C_o-semigroup on C^1, in particular $\|S_t'\| \leq Me^{\omega t}$ for

suitable M and ω.

We show now that $(T_t)_{t\geq 0}$, $T_t := e^{-\omega t}S_t + \omega \int_0^t e^{-\omega s}S_s\, ds$, is an integrated semigroup that has $A - \omega$ as the L-generator and fulfills 4.2(i).

Obviously $(T_t)_{t\geq 0}$ is exponentially bounded.

For large enough λ we have

$$\begin{aligned}\lambda\int_0^\infty e^{-\lambda t}T_t\, dt &= \lambda\int_0^\infty e^{-(\lambda+\omega)t}S_t\, dt + \omega\int_0^\infty e^{-(\lambda+\omega)t}S_t\, dt \\ &= (\lambda+\omega)\int_0^\infty e^{-(\lambda+\omega)}S_t\, dt \\ &= (\lambda + \omega - A)^{-1}.\end{aligned}$$

By remark 3.3 $(T_t)_{t\geq 0}$ is an integrated semigroup with L-generator $A - \omega$.

We show that $(T_t)_{t\geq 0}$ is Lipschitz-continuous.

Denote the first differentiation space of $(S_t)_{t\geq 0}$ (resp. $(T_t)_{t\geq 0}$) by C_S^1 (resp. C_T^1). From the representation of $(T_t)_{t\geq 0}$ we see $C_S^1 \subset C_T^1$.

From the second diagram in section 3 we derive

$D(A) \subset C_S^1 \subset C_T^1 \subset \overline{D(A-\omega)} = \overline{D(A)}$.

Since, as we mentioned before, C_S^1 is closed we obtain $C_S^1 = C_T^1$. On $C_T^1 = C_S^1$ we have

$\|T_t'\| = \|-\omega e^{-\omega t}S_t + e^{-\omega t}S_t' + \omega e^{-\omega t}S_t\| = \|e^{-\omega t}S_t'\| \leq M$.

The representation of $(T_t)_{t\geq 0}$ implies that $(T_t)_{t\geq 0}$ is locally Lipschitz-continuous. Let L be the local Lipschitz constant for the t-interval $[0,1]$.

Then ML is the global Lipschitz constant for $(T_t)_{t\geq 0}$.

To see this let $0 \leq s < t$.

Choose a partition $s = t_0 < t_1 < \ldots < t_n = t$ such that $t_{i+1} - t_i \leq 1$. Then

$$\|T_t - T_s\| \leq \Sigma_{i=0}^{n-1} \|T_{t_{i+1}} - T_{t_i}\|$$
$$= \Sigma_{i=0}^{n-1} \|T'_{t_i} T_{t_{i+1}-t_i}\|$$

(This is the functional equation 1.1(3) differentiated with respect to s.)

$$\leq ML \Sigma_{i=0}^{n-1} |t_{i+1} - t_i| = ML|t - s|.$$

By theorem 4.2

$$\|(\mu - A + \omega)^{-n}\| \leq ML/\mu^n \quad \text{for } n \geq 1 \text{ and } \mu > 0.$$

or $\|(\lambda - A)^{-n}\| \leq ML/(\lambda - \omega)^n$ for $n \geq 1$ and $\lambda > \omega$.

(ii) => (i):

(ii) implies

$$\|(\mu - A + \omega)^{-n}\| \leq ML/\mu^n \quad \text{for } n \geq 1 \text{ and } \mu > 0.$$

Therefore by theorem 4.2 $A - \omega$ is the L-generator of an integrated semigroup $(T_t)_{t\geq 0}$ with $\|T_t - T_s\| \leq M|t - s|$ for $s,t \geq 0$. Define $(S_t)_{t\geq 0}$ by

$$S_t := e^{\omega t} T_t - \omega \int_0^t e^{\omega s} T_s \, ds.$$

$(S_t)_{t\geq 0}$ is locally Lipschitz-continuous, exponentially bounded and $\lambda \int_0^\infty e^{-\lambda t} S_t \, dt = (\lambda - A)^{-1}$ for $\lambda > \omega$ (which is seen as before). Therefore, by remark 3.3, $(S_t)_{t\geq 0}$ is an integrated semigroup with L-generator A.

The following theorem gives a necessary condition for arbitrary L-generators:

Theorem 4.4 (Arendt):

Let $A:D(A) \to E$ be L-generator of an integrated semigroup $(S_t)_{t\geq 0}$ with $\|S_t\| \leq Me^{\omega t}$ for $t \geq 0$. Then $(\omega, \infty) \subset \rho(A)$ and for $n \in \mathbb{N} \cup \{0\}$ and $\lambda > \omega$:

$$\|(\frac{d}{d\lambda})^n (R_\lambda / \lambda)\| \leq n! M/(\lambda - \omega)^{n+1}.$$

Remark 4.5:

Two properties of R_λ follow from this that will be useful later:

There is a $c \geq 0$ such that: $\|R_\lambda\| \leq c$ for $\lambda > \omega$.

There is a $c' \geq 0$ such that:

$$\|R_\lambda^2\| \leq \frac{c'}{(\lambda - \omega)} \quad \text{for} \quad \lambda > \omega.$$

The following theorem shows that the resolvent estimates characterize at least the densely defined L-generators.

Theorem 4.6 (Arendt-Neubrander [4],[21]):

Let $A:D(A) \to E$ be densely defined.

Equivalent are:

(i) A is an L-generator.

(ii) There are real constants M, ω such that

$(\omega, \infty) \subset \rho(A)$ and for $n \in \mathbb{N}\cup\{0\}$ and $\lambda > \omega$

$$\|(\frac{d}{d\lambda})^n (R_\lambda/\lambda)\| \leq n!M/(\lambda - \omega)^{n+1} .$$

Remark 4.7

1) Condition (ii) was also used by M. Sova to characterize certain sorts of solvability of abstract Cauchy problems ([25], theorems 2.1 and 2.2). The condition he used was of a more general form, namely: For $q \in \mathbb{N}\cup\{0\}$

there are real constants M, ω such that

(q) $(\omega, \infty) \subset \rho(A)$ and for $n \in \mathbb{N}\cup\{0\}$ and $\lambda > \omega$

$$q!\,\|(\frac{d}{d\lambda})^n (R_\lambda/\lambda^q)\| \leq n!M/(\lambda - \omega)^{n+1} .$$

Although Sova's result remains on the level of solutions certain types of operator families shine through, namely

q-times integrated semigroups.

Within this terminology his result reads as follows:

Let A be densely defined and $q \in \mathbb{N}\cup\{0\}$.

A is the L-generator of a q-times integrated semigroup iff (q) holds.

An explicit proof of this is given in [21]. Sova further showed how (q) is linked with distribution semigroups ([25], theorem 3.2). This result translates to:

Let A be densely defined.

A is the generator of an exponentially bounded distribution semigroup iff for some $q \in \mathbb{N}\cup\{0\}$ A is the L-generator of a q-times integrated semigroup.

This theorem shows that the most relevant distribution semigroups are already covered by q-times integrated semigroups.

2) W. Arendt was able to make also a statement about non-densely defined operators that fulfill these resolvent estimates ([4]): They are L-generators of twice integrated semigroups.

Section 5. Some classes of integrated semigroups

Invertible integrated semigroups

The following theorem shows that the invertible integrated semigroups do not form the most interesting class.

Theorem 5.1:

Let $(S_t)_{t\geq 0}$ be an integrated semigroup.

Equivalent are:

(i) For some $t_o > 0$ S_{t_o} is invertible.

(ii) There is an operator $A \in \mathcal{L}(E)$ such that

$$S_t = \int_o^t e^{sA}\, ds \quad \text{for } t \geq 0.$$

Proof:

(i) => (ii): From the functional equation we obtain

$$S_t = \int_o^t (S_{r+t} - S_r)\, dr\, S_t^{-1}.$$

From this $C^1 = E$ and $S_o' = I$ follow. Therefore $(S_t')_{t\geq 0}$ is a C_o-semigroup on E with a generator A or $S_t = \int_o^t e^{sA} ds$. This implies: $S_{t_o} E \subset D(A)$. Since S_{t_o} is surjective $D(A) = E$ and $A \in \mathcal{L}(E)$.

(ii) => (i): Let $S_t = \int_o^t e^{sA} ds$ with some $A \in \mathcal{L}(E)$. $\int_o^t e^{sA} ds$ is invertible for small enough t because

$$\int_o^t e^{sA} ds = t - \int_o^t (e^{sA} - I)\, ds = t(I - \tfrac{1}{t}\int_o^t (e^{sA} - I)\, ds).$$

So the proof is complete if $\|\frac{1}{t}\int_o^t (e^{sA} - I)\, ds\| \to 0$ for $t \to 0$. This can be shown in a short calculation.

Bounded derivatives

Theorem 5.2:

Let $(S_t)_{t\geq 0}$ be an integrated semigroup on E. If for some $t_o > 0$ the operator $S'_{t_o} : C^1 \to C^1$ is bounded with respect to $\| \ \|_E$ then $(S_t)_{t\geq 0}$ is exponentially bounded.

Proof:

Let, for simplicity, $t_o = 1$.

From $S_{t+1} = S'_1 S_t + S_1$ we obtain

$$\|S_{t+1}\| \leq \|S'_1\| \|S_t\| + \|S_1\| .$$

With this one can show by induction

$$\|S_t\| \leq (\sup_{0\leq r\leq 1} \|S_r\|) \, e^{\|S'_1\| t} .$$

Corollary 5.3:

Let $(S_t)_{t\geq 0}$ be a locally Lipschitz-continuous integrated semigroup on E.

Then

(i) C^1 is closed with respect to $\| \ \|_E$.

(ii) $(S_t)_{t\geq 0}$ is exponentially bounded.

Proof:

(i) follows by theorem 2.5 and (ii) by theorem 5.2.

A look at the second diagram in section 3 shows: Instead of (i) one could also say: $C^1 = \overline{D(A)}$.

Remark 5.4:

Corollary 5.3 shows in particular that integrated semigroups that are locally Lipschitz-continuous and non-degenerate have an L-generator.

Finite dimensional spaces

Theorem 5.5:

Let $(S_t)_{t\geq 0}$ be an integrated semigroup on a finite dimensional vector space E.
Then $S_t = \int_0^t e^{sA}\, ds\, P$ for some $A \in L(E)$ and a projection P that commutes with A.

Proof:

S_1' is, as an operator on the finite dimensional space C^1, bounded. According to theorem 5.2 $(S_t)_{t\geq 0}$ is exponentially bounded. Therefore the Laplace integral exists for large enough λ. By the dimension formula

$$\dim(\ker R_\lambda) + \dim(R_\lambda E) = \dim(E) .$$

$\ker R_\lambda = \ker S_0'$ (prop. 3.2) and $R_\lambda E = S_0' R_\lambda E$ (prop. 3.4 (iii)) imply that $\ker R_\lambda$ and $R_\lambda E$ are complementary subspaces - as kernel and range of a projection. The direct sum $\ker R_\lambda \oplus R_\lambda E$ has therefore the dimension of the whole space E. And so $\ker R_\lambda \oplus R_\lambda E = E$. Since both spaces are contained in C^1 one obtains $C^1 = E$.
On $C^1 = E$ we have $S_s' S_t' = S_{s+t}'$.
$S_0' E$ is invariant under $(S_t')_{t\geq 0}$ and $S_0'|_{S_0' E} = I_{|S_0' E}$.
Thus by $T_t := S_t'|_{S_0' E}$ a C_0-semigroup is given on $S_0' E$.

Since $S''_t = S''_o S'_t$ on E and $S''_o S'_o E = S'_o S''_o E \subset S'_o E$ $C := S''_{o|S'_o E}$ is the generator of $(T_t)_{t\geq 0}$ or $T_t = e^{tC}$. Therefore $S'_t = S'_t S'_o = e^{tC} S'_o = e^{tS''_o} S'_o$. This implies the assertion with $A := S''_o$ and $P := S'_o$.

Integrated Groups

Definition 5.6:

A family $(S_t)_{t\in \mathbb{R}}$ of operators $S_t \in L(E)$ is called an integrated group if

1) $S_o = 0$,

2) $t \to S_t$ is strongly continuous,

3) $S_s S_t = \int_o^s (S_{r+t} - S_r)dr$ for $s,t \in \mathbb{R}$.

Remark 5.7:

An integrated group is composed of two integrated semigroups namely $(S_t)_{t\geq 0}$ and $(-S_{-t})_{t\geq 0}$.
Examples of integrated groups are integrated C_o-groups.
In the following we shall restrict ourselves to exponentially bounded integrated groups, i.e. integrated groups $(S_t)_{t\in \mathbb{R}}$ that fulfill $\|S_t\| \leq Me^{\omega|t|}$ for $t \in \mathbb{R}$ (M, ω are suitable constants).

Theorem 5.8:

Let $(S_t)_{t\in\mathbb{R}}$ be a strongly continuous family of operators $S_t \in L(E)$ such that $S_o = 0$, $\|S_t\| \leq Me^{\omega|t|}$ for $t\in\mathbb{R}$ and

$$R_\lambda := \begin{cases} \lambda\int_o^\infty e^{-\lambda t}S_t\,dt & \text{for } \lambda > \omega \\ -\lambda\int_o^\infty e^{\lambda t}S_{-t}\,dt & \text{for } \lambda < -\omega. \end{cases}$$

Equivalent are:

(i) $(S_t)_{t\in\mathbb{R}}$ is an integrated group.

(ii) $(R_\lambda)_{\lambda\in(-\infty,-\omega)\cup(\omega,\infty)}$ is a pseudo-resolvent.

Proof:

Remark 3.3 yields:

The functional equation holds for $0 \leq s,t$ and $s,t \leq 0$ if and only if $(R_\lambda)_\lambda$ fulfills the resolvent equation for $\lambda,\mu > \omega$ and $\lambda,\mu < -\omega$.

So the proof is complete if we can show:

The functional equation holds for $s \leq 0 \leq t$ if and only if $(R_\lambda)_\lambda$ fulfills the resolvent equation for $\mu < -\omega \leq \omega < \lambda$.

This can be shown by imitating the proof of remark 3.3 with minor changes (see [3], p.83).

Corollary 5.9:

Equivalent are:

(i) $A:D(A) \to E$ is L-generator of an integrated group $(S_t)_{t\in\mathbb{R}}$ such that $\|S_t\| \leq Me^{\omega|t|}$ for $t\in\mathbb{R}$ ($M,\omega \geq 0$ suitable constants).

(ii) A and $-A$ are L-generators.

Proof:

(i) => (ii): We have to show: $-A$ is an L-generator.

By theorem 5.8 $(-\infty,-\omega) \subset \rho(A)$ and

$$(\lambda - A)^{-1} = -\lambda\int_0^\infty e^{\lambda t} S_{-t}\,dt \quad \text{for} \quad \lambda < -\omega.$$

Therefore

$$(\mu + A)^{-1} = \mu\int_0^\infty e^{-\mu t}(-S_{-t})\,dt \quad \text{for} \quad \mu > \omega,$$

i.e. $-A$ is L-generator of $(-S_{-t})_{t\geq 0}$.

(ii) => (i): Let A be L-generator of $(U_t)_{t\geq 0}$ and $-A$ L-generator of $(V_t)_{t\geq 0}$.

The latter statement implies $(-\infty,-\omega) \subset \rho(A)$ for some $\omega \geq 0$ and

$$(\lambda - A)^{-1} = -\lambda\int_0^\infty e^{\lambda t}(-V_t)\,dt \quad \text{for} \quad \lambda < -\omega.$$

The family $(S_t)_{t\in\mathbb{R}}$, $S_t := \begin{cases} U_t & , t \geq 0 \\ -V_{-t} & , t < 0 \end{cases}$, now is according to theorem 5.8 an integrated group.

Section 6. Perturbations

In this section we shall be concerned with (additive) perturbations of L-generators.

As the applications (part II) will show the differentiation spaces C^n play an important role. Therefore we shall also consider the differentiation spaces that belong to the perturbed and unperturbed integrated semigroups.

We shall choose the following notation:

If $A + B$ is the L-generator of the integrated semigroup $(S_t^B)_{t\geq 0}$ then

$C_B^n := \{ x \in E : t \to S_t^B x \text{ is n-times cont. differentiable}\}$.

The n-th differentiation space of the unperturbed integrated semigroup will always be denoted by C^n .

The new and surprising situation, compared to C_o-semigroups and cosine functions, is that even bounded perturbations may fail to preserve the property "L-generator".

Example 6.1:

In part II, section 2 we shall show: If an operator A generates a cosine-function then $\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ is an L-generator on $E \times E$. The bounded perturbation $\begin{pmatrix} 0 & -I \\ 0 & 0 \end{pmatrix}$ leads to $\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$ $=: \hat{A}$. But this operator is never an L-generator if A is unbounded since then $\rho(\hat{A}) = \emptyset$.

Theorem 6.2 (commuting perturbations):

Let $A:D(A) \to E$ be L-generator of an integrated semigroup $(S_t)_{t\geq 0}$.

Let further B be a bounded operator that commutes with A (i.e. $BD(A) \subset D(A)$ and $BA = AB$).

Then $A + B$ is L-generator of the integrated semigroup $(S_t^B)_{t\geq 0}$, given by

$$S_t^B = e^{tB}S_t - B\int_0^t e^{sB}S_s\,ds$$

Further: $C_B^n = C^n$ for $n \geq 1$.

Proof:

For certain constants M, ω we have $\|S_t\| \leq Me^{\omega t}$ and also $\|S_t^B\| \leq Me^{\omega t}$ for $t \geq 0$.

Let, for $\lambda > \omega$,

$$R_\lambda^B := \lambda\int_0^\infty e^{-\lambda t}S_t^B\,dt \quad \text{and} \quad R_\lambda := (\lambda - A)^{-1}.$$

We show that $R_\lambda^B = (I - R_\lambda B)^{-1}R_\lambda$ for large enough λ.

$I - R_\lambda B$ is invertible for large enough λ since

$\|(R_\lambda B)^2\| = \|R_\lambda{}^2 B^2\| \leq \|B^2\|\, c/(\lambda - \omega)$ (remark 4.5).

The right hand side can be transformed:

$$(I - R_\lambda B)^{-1}R_\lambda = \Sigma_{i=0}^\infty B^i R_\lambda{}^{i+1}$$

$$= \Sigma_{i=0}^\infty B^i (-1)^i \frac{1}{i!} \left(\frac{d}{d\lambda}\right)^i R_\lambda$$

$$= \Sigma_{i=0}^\infty B^i (-1)^i \frac{1}{i!} \left(\frac{d}{d\lambda}\right)^i \left(\lambda\int_0^\infty e^{-\lambda t}S_t\,dt\right)$$

$$= \Sigma_{i=0}^\infty B^i (-1)^i \frac{1}{i!} \lambda\left(\frac{d}{d\lambda}\right)^i \int_0^\infty e^{-\lambda t}S_t\,dt$$

$$+ \Sigma_{i=0}^\infty B^i (-1)^i \frac{1}{i!}\, i\left(\frac{d}{d\lambda}\right)^{i-1} \int_0^\infty e^{-\lambda t}S_t\,dt$$

$$= \lambda \Sigma_{i=0}^{\infty} B^i \frac{1}{i!} \int_0^{\infty} t^i e^{-\lambda t} S_t \, dt$$

$$- \Sigma_{i=1}^{\infty} B^i \frac{1}{(i-1)!} \int_0^{\infty} t^{i-1} e^{-\lambda t} S_t \, dt$$

$$= \lambda \int_0^{\infty} e^{-\lambda t} e^{tB} S_t \, dt - \int_0^{\infty} e^{-\lambda t} B e^{tB} S_t \, dt$$

$$= \lambda \int_0^{\infty} e^{-\lambda t} e^{tB} S_t \, dt - \lambda \int_0^{\infty} e^{-\lambda t} B \int_0^t e^{sB} S_s \, ds \, dt$$

$$= R_\lambda^B$$

Since $(I - R_\lambda B)^{-1} R_\lambda = \Sigma_{i=0}^{\infty} (R_\lambda B)^n R_\lambda$

$= R_\lambda \Sigma_{i=0}^{\infty} (R_\lambda B)^i = R_\lambda (I - B R_\lambda)^{-1}$

we also have $R_\lambda^B = R_\lambda (I - B R_\lambda)^{-1}$.

A short calculation now shows: R_λ^B is the resolvent of $A + B$. Together with remark 3.3 this implies: $(S_t^B)_{t \geq 0}$ is an integrated semigroup and $A + B$ is its L-generator.

Proof of the additional assertion:

$C^n \subset C_B^n$ is a direct consequence of the representation of $(S_t^B)_{t \geq 0}$.

$C_B^n \subset C^n$ follows from the fact that also $(S_t)_{t \geq 0}$ can be represented as $S_t = e^{-tB} S_t^B + B \int_0^t e^{-sB} S_s^B \, ds$ ($-B$ commutes with $A + B$!).

Remark 6.3:

A consequence of this theorem is: If A is L-generator of the integrated semigroup $(S_t)_{t \geq 0}$ then for any $\mu \in \mathbb{C}$ $A + \mu$ is the L-generator of the integrated semigroup $(S_t^\mu)_{t \geq 0}$ where

$$S_t^\mu = e^{\mu t} S_t - \mu \int_0^t e^{\mu s} S_s \, ds .$$

Theorem 6.4:

Let $A:D(A) \to E$ be an L-generator.

Let $B \in L(C^1)$ (C^1 being equipped with $\| \ \|_E$) with the property: There are M, ω such that

$$\|S_t'B\| \leq Me^{\omega t} \quad \text{for } t \geq 0.$$

Then $A + B$ is an L-generator.

Proof:

Extend $S_t'B$ boundedly to an operator $\overline{S_t'B}$ on $C^1 = \overline{D(A)}$. The mapping $t \to \overline{S_t'B}$ is, by the assumption, strongly continuous on $\overline{D(A)}$.

Consider now the integral equation

$$(*) \qquad S_t^B = S_t + \int_0^t \overline{S_{t-r}'B}\, S_r^B \, dr.$$

Let $c > 0$ and equip the space $F_c := C([0,c], L_s(E, \overline{D(A)}))$ with the norm $\|T\| := \sup_{0 \leq t \leq c} \|T_t\|_{L(E)}$. F_c is an Banach space.

By $(JT)(t) := \int_0^t \overline{S_{t-r}'B}\, T_r \, dr$ a bounded operator is given on F_c.

The integral equation can now be written in the form

$$(I - J)S^B = S,$$

where $S^B := (S_t^B)_{0 \leq t \leq c}$ and $S := (S_t)_{0 \leq t \leq c}$ ($S \in F_c$ since $S_t E \subset C^1$).

By an estimate one can show: The spectral radius of this operator $r(J) = 0$.

Thus the operator $I - J$ is invertible, i.e. there is exactly one solution of (*): $S^B \in F_c$.

For different c and c' the solutions coincide on $[0,c] \cap [0,c']$ (by uniqueness). One obtains therefore a unique solution of (*) on $\mathbb{R}_+$.

We uncover this solution as the "perturbed" integrated semigroup.

From the integral equation (*) we get the estimate

$$\|S_t^B\|_{L(E)} \leq M' e^{\omega' t} \quad \text{for } t \geq 0$$

(with suitable constants M', ω').

Define for $\lambda > \omega'$

$$R_\lambda^B := \lambda \int_0^\infty e^{-\lambda t} S_t^B dt.$$

Then we have:

a) $R_\lambda^B E \subset \overline{D(A)}$ (since $S_t^B E \subset \overline{D(A)}$. This holds by $S^B \in F_c$.).

b) The assumption implies for $\lambda > \omega$:

$$\|R_\lambda B\| \leq M/(\lambda - \omega) \qquad (R_\lambda := (\lambda - A)^{-1})$$

since $R_\lambda B = \lambda \int_0^\infty e^{-\lambda t} S_t B \, dt$

$$= \int_0^\infty e^{-\lambda t} S_t' B \, dt.$$

Therefore $R_\lambda B$ can be extended to $\overline{D(A)}$ and

b') $\|\overline{R_\lambda B}\| \leq M/(\lambda - \omega)$.

a) shows that $\overline{R_\lambda B} \, R_\lambda^B$ is a legitimate term. We further have:

$$\overline{R_\lambda B} \, R_\lambda^B = \int_0^\infty e^{-\lambda t} \, \overline{S_t' B} \, dt \, R_\lambda^B$$

$$= \lambda \int_0^\infty \int_0^\infty e^{-\lambda(t+r)} \, \overline{S_t' B} \, dt \, S_r^B \, dr$$

$$= \lambda \int_0^\infty \int_r^\infty e^{-\lambda u} \, \overline{S_{u-r}' B} \, du \, S_r^B \, dr$$

(substituting by $u := t + r$)

$$= \lambda \int_0^\infty e^{-\lambda u} \int_0^u \overline{S_{u-r}' B} \, S_r^B \, dr du$$

$$= \lambda \int_0^\infty e^{-\lambda u} \, S_u^B \, du - \lambda \int_0^\infty e^{-\lambda u} \, S_u \, du$$

$$= R_\lambda^B - R_\lambda.$$

By b') $I - R_\lambda B$ is invertible and for sufficiently large λ we obtain from the above equation

$$R_\lambda^B = (I - R_\lambda B)^{-1} R_\lambda .$$

Moreover

$$\begin{aligned} R_\lambda^B &= \Sigma_{i=0}^{\infty} (R_\lambda B)^i R_\lambda \\ &= R_\lambda \Sigma_{i=0}^{\infty} (BR_\lambda)^i \qquad \text{(since B is bounded)} \\ &= R_\lambda (I - BR_\lambda)^{-1} . \end{aligned}$$

This implies that R_λ^B is the inverse of $\lambda - A - B$. And by remark 3.3 we finally obtain: $(S_t)_{t\geq 0}$ is an integrated semigroup and $A + B$ is its L-generator.

Theorem 6.5:

Let A be the L-generator of a locally Lipschitz-continuous integrated semigroup and $B \in \mathcal{L}(\overline{D(A)}, E)$ ($\overline{D(A)}$ equipped with $\| \ \|_E$). Then $A + B$ is the L-generator of a locally-Lipschitz-continuous integrated semigroup. Further $C_B^1 = C^1 = \overline{D(A)}$.

Proof:

By theorem 4.3 the proof reduces to a resolvent estimate. Let M, ω be such that (according to theorem 4.3)

$\|(\lambda - A)^{-n}\| \leq M/(\lambda - \omega)^n$ for $n \in \mathbb{N}$ and $\lambda > \omega$.

Then for $\lambda > M\|B\| + \omega$ we have (with $R_\lambda := (\lambda - A)^{-1}$)

$\|BR_\lambda\| \leq \|B\| M/(\lambda - \omega) < 1.$

Therefore $I - BR_\lambda$ is invertible for such λ and

$R_\lambda (I - BR_\lambda)^{-1} = (\lambda - A - B)^{-1}$.

Let $n \in \mathbb{N}$. Then

$$\|(\lambda - A - B)^{-n}\| = \|(R_\lambda(I - BR_\lambda)^{-1})^n\|$$

$$= \|(R_\lambda \sum_{i=0}^{\infty} (BR_\lambda)^i)^n\|$$

$$= \|\sum_{j=0}^{\infty} \sum_{i_1+\ldots+i_n=j} R_\lambda(BR_\lambda)^{i_1} \ldots R_\lambda(BR_\lambda)^{i_n}\|$$

$$\leq \sum_{j=0}^{\infty} \sum_{i_1+\ldots+i_n=j} \|R_\lambda(BR_\lambda)^{i_1} \ldots R_\lambda(BR_\lambda)^{i_n}\| .$$

Each product $\|R_\lambda(BR_\lambda)^{i_1} \ldots R_\lambda(BR_\lambda)^{i_n}\|$ is of the form

$$\|R_\lambda^{k_1} B R_\lambda^{k_2} \ldots B R_\lambda^{k_m}\| \qquad (*)$$

which can be estimated by

$$\leq M(\tfrac{1}{\lambda-\omega})^{k_1} \|B\| M(\tfrac{1}{\lambda-\omega})^{k_2} \ldots \|B\| M(\tfrac{1}{\lambda-\omega})^{k_m}$$

$$= M(\tfrac{1}{\lambda-\omega})^{k_1} (\|B\|M)(\tfrac{1}{\lambda-\omega})^{k_2} \ldots (\|B\|M)(\tfrac{1}{\lambda-\omega})^{k_m}.$$

A comparison of this product and (*) shows that we have

$$= M \tfrac{1}{\lambda-\omega} (\tfrac{\|B\|M}{\lambda-\omega})^{i_1} \ldots \tfrac{1}{\lambda-\omega} (\tfrac{\|B\|M}{\lambda-\omega})^{i_n}.$$

The whole series can now be estimated by

$$\leq M \sum_{j=0}^{\infty} \sum_{i_1+\ldots+i_n=j} \tfrac{1}{\lambda-\omega} (\tfrac{\|B\|M}{\lambda-\omega})^{i_1} \ldots \tfrac{1}{\lambda-\omega} (\tfrac{\|B\|M}{\lambda-\omega})^{i_n}.$$

Going back the same way we obtain

$$= M/(\lambda - \omega - M\|B\|)^n.$$

Together with theorem 4.3 this implies the assertion. The additional assertion follows from remark 2.4 and from the fact that locally-Lipschitz-continuous integrated semigroups have locally bounded derivatives.

The following lemma will be needed in the proof of theorem 6.7. It will be useful again in part II.

Lemma 6.6 (Arendt [3] prop. 3.5):

Let A L-generator of an integrated semigroup $(S_t)_{t\geq 0}$.

Then for all $x \in E$ and $t \geq 0$

$\int_0^t S_s x \, ds \in D(A)$ and $S_t x = A\int_0^t S_s x \, ds + tx$.

Theorem 6.7:

Let A be an L-generator and $B \in \mathcal{L}(E)$.

Suppose there is a $c \geq 0$ such that

$\|BAx\| \leq c\|x\|$ for all $x \in D(A)$.

Then $A + B$ is an L-generator.

Further: $C^1 \subset C_B^1$.

Proof:

The assumption implies:

BA can be extended boundedly to $\overline{D(A)}$ and

$\|\overline{BA}\|_{\mathcal{L}(\overline{D(A)},E)} \leq c$.

Consider now the integral equation

$$S_t^B = S_t + \int_o^t S_{t-r}^B(\overline{BA}S_r + B)\,dr.$$

The integration kernel fulfills $\overline{BA}S_r + B \in \mathcal{L}(E)$ since $S_s E \subset \overline{D(A)}$ and $r \to \overline{BA}S_r + B$ is strongly continuous.

As in the proof of theorem 6.4 one can show:

The integral equation is uniquely solvable on $\mathbb{R}_+$.

For the solution $(S_t^B)_{t\geq 0}$ the following holds

$$\int_o^t S_s^B\,ds - \int_o^t S_s\,ds$$

$$= \int_o^t\int_o^s S_{s-r}^B(\overline{BA}S_r + B)\,dr\,ds$$

$$= \int_o^t\int_o^s S_r^B(\overline{BA}\,S_{s-r} + B)\,dr\,ds$$

$$= \int_o^t\int_r^t S_r^B(\overline{BA}\,S_{s-r} + B)\,ds\,dr$$

$$= \int_o^t S_r^B(\overline{BA}\int_o^{t-r} S_s ds + (t-r)B)\,dr$$

$$= \int_o^t S_r^B\,B\,S_{t-r}\,dr \quad \text{(by lemma 6.6)}.$$

$(S_t^B)_{t\geq 0}$ fulfills for certain M, ω the estimate

$\|S_t^B\| \leq Me^{\omega t}$ for $t \geq 0$.

Define for $\lambda > \omega$ $R_\lambda^B := \lambda\int_0^\infty e^{-\lambda t}S_t^B\,dt$.

Using the above equation one can show like in the proof of theorem 6.4 that

$$R_\lambda^B BR_\lambda = R_\lambda^B - R_\lambda \quad (R_\lambda := (\lambda - A)^{-1}).$$

Assume without loss of generality that $\|S_t\| \leq Me^{\omega t}$ for $t \geq 0$ and fix some $\lambda_0 > \omega$. Then for $\lambda > \lambda_0$

$$\begin{aligned}\|R_\lambda R_{\lambda_0}\| &= \|(R_\lambda - R_{\lambda_0})/(\lambda - \lambda_0)\| \\ &\leq (\|R_\lambda\| + \|R_{\lambda_0}\|)/(\lambda - \lambda_0) \\ &\leq c/(\lambda - \lambda_0) \quad \text{(remark 4.5)}.\end{aligned}$$

From this we obtain

$$\|BR_\lambda\| = \|B(\lambda_0 - A)R_\lambda R_{\lambda_0}\| \leq \|B(\lambda_0 - A)\|c/(\lambda - \lambda_0).$$

Thus $I - BR_\lambda$ is invertible for large enough λ and

$$R_\lambda^B = R_\lambda(I - BR_\lambda)^{-1}.$$

This shows again:

$(R_\lambda^B)_{Re\lambda>\omega}$ is the resolvent of $A + B$, $(S_t)_{t\geq 0}$ is an integrated semigroup and $A + B$ its L-generator.

Proof of the additional assertion:

$(S_t^B)_{t\geq 0}$ fulfills, as solution of the integral equation, also

$$S_t^B = S_t + \int_0^t S_r^B(\widetilde{BA}S_{t-r} + B)\,dr$$

The right hand side is strongly continuously differentiable on C^1.

Section 7. A C_o-semigroup linked with $(S_t)_{t\geq 0}$

We showed earlier (remark 2.4) that $(S_t')_{t\geq 0}$ is not necessarily a C_o-semigroup on C^1. But if $(S_t)_{t\geq 0}$ is exponentially bounded and non-degenerate one can construct a normed subspace of C^1 such that $(S_t')_{t\geq 0}$, restricted to this space, gives a C_o-semigroup. This is what we shall do in the following.

We assume in this whole section:

$(S_t)_{t\geq 0}$ is non-degenerate and for certain M and ω

$$\|S_t\| \leq Me^{\omega t} \quad \text{for } t \geq 0.$$

(Thereby we assume without loss of generality $\omega \geq 0$ and $M \geq 1$. This will be used in the proof of theorem 7.1(i).)

We denote the L-generator of $(S_t)_{t\geq 0}$ by A.

Theorem 7.1:

(i) By $\|x\|_R := \sup_{t\geq 0} \|e^{-\omega t}S_t'x\|$ a norm is given on D(A) with the property

$$\|x\| \leq \|x\|_R \leq M\|x\|_1 \quad \text{for } x \in D(A).$$

(ii) Let F be the closure of C^2 with respect to this norm.

Then $F \subset C^1$ and by $(T_t)_{t\geq 0}$, $T_t := S_t'|_F$, a C_o-semigroup is given on $(F, \| \|_R)$.

(iii) If D(A) is dense in E then $D(A) \subset F$.

Proof:

(i): Let $x \in D(A)$. Then $S'_t x = S_t Ax + x$.

From this we obtain

$$\|S'_t x\| \leq \|S_t\| \|Ax\| + \|x\|$$

$$\leq Me^{\omega t}\|Ax\| + \|x\| \leq Me^{\omega t}\|x\|_1 \quad \text{and therefore (i).}$$

(ii): Let $x \in F$ and (x_n) a sequence in C^2, converging to x with respect to $\| \ \|_R$. By (i) $x_n \to x$ with respect to $\| \ \|_E$.

Let $c > 0$. Then $S_t x_n \to S_t x$ for $0 \leq t \leq c$.

Since (x_n) is a $\| \ \|_R$-Cauchy sequence $S'_t x_n \to g(t)$ uniformly for $0 \leq t \leq c$ for some function $g:[0,c] \to E$.

Therefore $t \to S_t x$ is a C^1-function on $[0,c]$. Since c is arbitrary $x \in C^1$.

According to prop. 2.2(ii) $S'_s C^2 \subset C^2$.

Furthermore we have for $x \in C^2$

$$\|S'_s x\|_R = \sup_{0 \leq t \leq \infty} \|e^{-\omega t} S'_t \, S'_s x\|$$

$$= e^{\omega s} \sup_{0 \leq t \leq \infty} \|e^{-\omega(t+s)} S'_{t+s} \, x\| \leq e^{\omega s} \|x\|_R$$

Thus $T_s = S'_s|_F \in L(F)$.

To prove the strong continuity of $(T_t)_{t \geq 0}$ let $x \in D(A^2)$.

$$\|(S'_t - I)x\|_R = \|S_t Ax\|_R \leq M\|S_t Ax\|_1 \quad \text{(by (i))}$$

$$= M(\|S_t A^2 x\| + \|S_t Ax\|).$$

If $t \to 0$ this converges to 0.

Since $D(A^2)$ is dense in C^2 with respect to $\| \ \|_1$ (see the first diagram in section 3) and therefore also with respect to $\| \ \|_R$ since further $(T_t)_{t \geq 0}$ is uniformly bounded on compact t-intervals (as we just showed) the strong continuity extends to the whole of F.

(iii): If $\overline{D(A)} = E$ then $D(A) = \overline{D(A^2)}^1 \subset \overline{D(A^2)}^R \subset F$.

Remarks:

1) In general the C_o-semigroup $(T_t)_{t\geq 0}$ does not determine the integrated semigroup $(S_t)_{t\geq 0}$. (See the integrated semigroups $(S_t^B)_{t\geq 0}$ in example 2.8.) This holds though if D(A) is dense in E.

2) The index R in $\| \ \|_R$ should refer to an application: In part II, section 3 this norm shall turn up when second-order-problems are reduced to first-order-problems. $\| \ \|_R$ can therefore be considered as a reduction norm.

The following theorem shows that this construction closely links the properties a) L-generator and b) generator.

Theorem 7.2:

Let $A:D(A) \to E$ be a densely defined operator.

Equivalent are:

(i) A is L-generator.

(ii) $\rho(A) \neq \emptyset$ and there exists a norm $\| \ \|_R$ on D(A) such that

1) $\| \ \|_E \leq \| \ \|_R \leq c\| \ \|_1$ (graph norm of A),

2) $A_{\overline{D(A)}^R}$ is the generator of a C_o-semigroup on $(\overline{D(A)}^R, \| \ \|_R)$.

($A_{\overline{D(A)}^R}$ is "the part of A in $\overline{D(A)}^R$", i.e. the operator $A_{|D}$, where $D := \{ x \in D(A) : Ax \in \overline{D(A)}^R \}$.)

Proof:

(i) => (ii) has been shown already in theorem 7.1.

(ii) => (i) follows from the lemmas 7.3 and 7.4.

Lemma 7.3:

Let $A:D(A) \to E$ be a closed operator and

$\| \ \|_R$ a norm on $D(A)$ such that

1) $\| \ \|_E \leq \| \ \|_R \leq c\| \ \|_1$,

2) $A_{\overline{D(A)}^R}$ is the generator of a C_o-semigroup on $(\overline{D(A)}^R, \| \ \|_R)$.

Then $A_{|D(A^2)}$ is an L-generator on $(D(A), \| \ \|_1)$.

Proof:

Let $(T_t)_{t\geq 0}$ be the C_o-semigroup on $F := \overline{D(A)}^R$ generated by $A_{\overline{D(A)}^R}$. By $S_t := \int_o^t T_s \, ds$ an integrated semigroup is given on F. We show that $(S_{t|D(A)})_{t\geq 0}$ is an exponentially bounded integrated semigroup on $(D(A), \| \ \|_1)$.

Let $x \in D(A)$.

$$\begin{aligned} \|S_t x\|_1 &= \|AS_t x\| + \|S_t x\| \\ &\leq \|AS_t x\|_R + \|S_t x\|_R \\ &= \|(T_t - I)x\|_R + \|S_t x\|_R \\ &\leq M e^{\omega t} \|x\|_R \quad \text{for suitable } M, \omega \\ &\leq cMe^{\omega t} \|x\|_1 . \end{aligned}$$

Therefore $S_t \in L(D(A))$ and $(S_t)_{t\geq 0}$ is exponentially bounded..

$t \to S_t$ is strongly continuous with respect to $\| \ \|_1$ be-

cause:

$$\begin{aligned}\|(S_t - S_s)x\|_1 &= \|A\int_s^t T_r x\, dr\| + \|\int_s^t T_r x\, dr\| \\ &= \|(T_t - T_s)x\| + \|\int_s^t T_r x\, dr\| \\ &\leq \|(T_t - T_s)x\|_R + \|\int_s^t T_r x\, dr\|_R \\ &\to 0 \quad \text{as} \quad t \to s\ .\end{aligned}$$

Thus $(S_t)_{t\geq 0}$ is an exponentially bounded integrated semigroup on $D(A)$. On F we have $(\lambda - A_F)^{-1} = \lambda\int_0^\infty e^{-\lambda t}S_t\, dt$ (for λ sufficiently large). This implies the same on $D(A) \subset F$. Since $(S_t)_{t\geq 0}$ is exponentially bounded on $D(A)$ the Laplace integral maps $D(A)$ into $D(A)$. Therefore $(\lambda - A_F)^{-1}D(A) \subset D(A)$. Hence $A_{|D(A^2)}$ is the L-generator of $(S_t)_{t\geq 0}$.

Lemma 7.4:

Let $A:D(A) \to E$. Equivalent are:

(i) A is L-generator of an integrated semigroup on E.

(ii) $\rho(A) \neq \emptyset$ and $A_{|D(A^2)}$ is L-generator of an integrated semigroup on the Banach space $(D(A), \| \|_1)$.

Proof:

(i) => (ii) is clear.

(ii) => (i): Let $A_1 := A_{|D(A^2)}$ be the L-generator of the integrated semigroup $(S_t)_{t\geq 0}$. For $\lambda_0 \in \rho(A)$ the operator $T := \lambda_0 - A$ is an isomorphism from $(D(A), \| \|_1)$ onto $(E, \| \|_E)$. Therefore $(TS_tT^{-1})_{t\geq 0}$ is an integrated semigroup on E with L-generator TA_1T^{-1}. Obviously $TA_1T^{-1} = A$. This shows the assertion.

The lemmas 7.3 and 7.4 are also applied in the proof of

Theorem 7.5:

Let $A:D(A) \to E$ be L-generator of an integrated group $(S_t)_{t\in\mathbb{R}}$ with $\|S_t\| \leq Me^{\omega|t|}$ for $t\in\mathbb{R}$ (M,ω suitably chosen).

Then $A^2:D(A^2) \to E$ is L-generator of an integrated semigroup.

Proof:

We proceed like this:

First we construct a space $(F,\|\ \|_R)$ as in the proof of theorem 7.1.

Then we show that A_F^2 generates a C_o-semigroup. And with the lemmas 7.3 and 7.4 we go back into the space E.

We equip the space

$C_{\mathbb{R}}^2 := \{ x \in E : (S_t x)_{t\in\mathbb{R}}$ is twice cont. differentiable$\}$

with the norm $\|\ \|_R$, $\|x\|_R := \sup_{t\in\mathbb{R}} \|e^{-\omega|t|} S_t' x\|$.

Like in the proof of theorem 7.1 one can verify:

1) $\|\ \|_E \leq \|\ \|_R \leq c\|\ \|_1$.
2) The family $(S_t')_{t\in\mathbb{R}}$ leaves $C_{\mathbb{R}}^2$ invariant.
3) $\|S_t' x\|_R \leq e^{\omega|t|} \|x\|_R$ for $t\in\mathbb{R}$ and $x \in C_{\mathbb{R}}^2$.
4) $t \to S_t' x$ is continuous with respect to $\|\ \|_R$ for all $x \in C_{\mathbb{R}}^2$.
5) $S_o' = I$ on $C_{\mathbb{R}}^2$.
6) $S_s' S_t' = S_{s+t}'$ for $s,t \in \mathbb{R}$.
7) Let F be the $\|\ \|_R$-closure of $C_{\mathbb{R}}^2$. Then $(S_t)_{t\in\mathbb{R}}$ is strongly continuously differentiable on F.

2) - 7) imply:

$T_t := S'_t|_F$ gives a C_o-group on $(F, \| \|_R)$.

$(T_t)_{t\geq 0}$ has the generator A_F (the part of A in F).

From this we obtain that $(A_F)^2 = (A^2)_F$ is the generator of a C_o-semigroup on F.

The rest of the proof consists in showing that $A^2 : D(A^2) \to E$ fulfills the assumptions of the lemmas 7.3 and 7.4.

a) $\| \|_1 \leq c \| \|_{A^2}$ (graph norm of A^2): Let $\lambda > \omega$.

$$\begin{aligned} \|Ax\| + \|x\| &\leq c' \|(\lambda - A)x\| \\ &\leq c'' \|(\lambda + A)(\lambda - A)x\| \\ &\quad \text{since } -\lambda \in (-\infty, -\omega) \subset \rho(A) \\ &= c'' \|(\lambda^2 - A^2)x\| \\ &\leq c(\|A^2 x\| + \|x\|). \end{aligned}$$

Together with 1) this implies:

a') $\| \|_E \leq \| \|_R \leq c \| \|_{A^2}$,

b) $F = \overline{D(A^2)}^R$.

The latter assertion can be seen like this:

First, notice that $D(A^2) \subset C^2_{\mathbb{R}} \subset C^2$. From this one obtains by the second diagram in section 3 the chain $D(A^2) \subset C^2_{\mathbb{R}} \subset C^2 \subset \overline{D(A^2)}^1$. Therefore $D(A^2)$ is $\| \|_A$-dense in $C^2_{\mathbb{R}}$ and by $\| \|_R \leq c \| \|_A$ (see 1)) also $\| \|_R$-dense in F.

c) $\rho(A^2) \neq \emptyset$, because: $(-\infty, -\omega) \cup (\omega, \infty) \subset \rho(A)$ (by theorem 5.8). Since $\lambda^2 - A^2 = (\lambda - A)(\lambda + A)$ we have $(\omega^2, \infty) \subset \rho(A^2)$.

With a'), b) and c) the assumptions of the lemmas 7.3 and 7.4 are fulfilled and the assertion follows.

Remark 7.6:

An inspection of this proof shows that one can strengthen the assertion of theorem 7.5:

The integrated semigroup generated by A^2 has an analytical extension on a sector around the positive real axis.

This holds because the result from C_o-semigroup theory that we used in the proof is stronger: $(A^2)_F$ even generates an analytical C_o-semigroup $(U_t)_{t\geq 0}$ on F.

Proceding like in the proof of lemma 7.3 one deduces from this: The integrated semigroup $(\int_o^t U_s\, ds)_{t\geq 0}$ on $(D(A^2), \| \|_{A^2})$ has an analytical extension on an open sector around $[0,\infty)$.

The isomorphism $\lambda - A^2 : (D(A^2), \| \|_{A^2}) \to (E, \| \|_E)$ (for $\lambda \in \rho(A^2)$) does the rest.

This result means in particular:

The integrated semigroup generated by A^2 can be written as $(\int_o^t V_s\, ds)_{t\geq 0}$, where $(V_t)_{t>0}$ is a semigroup (i.e. a strongly continuous family $(V_t)_{t>0}$ of bounded operators V_t with $V_s V_t = V_{s+t}$ for $s,t > 0$) and $\int_o^t V_s\, ds$ is the improper integral $\lim_{\alpha\to o} \int_\alpha^t V_s\, ds$.

Example 7.7:

$A = \frac{d}{dx}$ on $L^\infty(\mathbb{R})$ (distributional domain) is the L-generator of the integrated group $(S_t)_{t\in\mathbb{R}}$,

$S_t f(x) := \int_o^t f(x+s)\, ds$. Here the semigroup $(V_t)_{t>0}$ is of the form $V_t f(x) = (\frac{1}{4\pi t})^{1/2} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} f(y)\, dy$.

This shows that $(V_t)_{t>0}$ cannot always be extended to a C_o-semigroup.

Section 8. Integrated semigroups and abstract Cauchy problems

In this section we ask how integrated semigroups are connected with abstract Cauchy problems

$$(*) \qquad \begin{aligned} u'(t) &= Au(t) \\ u(0) &= x. \end{aligned}$$

The question is:
What sort of wellposedness is described by integrated semigroups?

We first clear up in what sense we use the word "solvability".

Definition 8.1:

Let $A:D(A) \to E$ be a linear (not necessarily densely defined) operator.
A solution of $u' = Au$ is a C^1-function $u:[0,\infty) \to E$ such that $u(t) \in D(A)$ and $u'(t) = Au(t)$ holds for $t \geq 0$.
Let $x \in D(A)$. Then (*) is solvable for x if there is a solution of $u' = Au$ such that $u(0) = x$.

We use the following sort of wellposedness:

Definition 8.2:

Let $A:D(A) \to E$ be linear and D a subspace of $D(A)$, dense in $D(A)$ with respect to $\| \ \|_E$.

(*) is wellposed on D in the integrated sense if

(i) (*) is solvable for every $x \in D$,

(ii) for every compact t-interval $I \subset [0,\infty)$ there is a constant K_I such that

$$\sup_{t \in I} \left\| \int_0^t u(s)\, ds \right\| \leq K_I \, \|u(0)\|$$

for all solutions of $u' = Au$.

Remark 8.3:

(ii) implies in particular uniqueness of the solutions for all initial values in $D(A)$.
So uniqueness holds for all possible initial values, while solvability is required only for initial values in D. Such a "decoupling" of existence and uniqueness is also typical for higher order Cauchy problems. An example and its connection to this wellposedness is presented in section 2 of part II: the complete Cauchy problem of second order.

How this wellposedness is related to integrated semigroups is partially described by the following theorem.

Theorem 8.4:

Let $A:D(A) \to E$ be linear and D a subset of $D(A)$, dense in $D(A)$ with respect to $\| \ \|_E$.

If (*) is wellposed on D in the integrated sense then there is an integrated semigroup $(S_t)_{t\geq 0}$ on $(\overline{D(A)}, \| \ \|_E)$ with the properties

(a) $D \subset C^2$ and $A_{|D} \subset S_0''$,

(b) $S_0'_{|D} = I_{|D}$.

Proof:

We denote the solution for the initial value x by $u(*,x)$. Let $t \geq 0$. Then by $x \to u(t,x)$ a linear mapping $U_t: D \to D(A)$ is given. By 8.2(ii) the operator $\int_0^t U_s \, ds: D \to \overline{D(A)}$ can be extended to a bounded linear operator $S_t: \overline{D(A)} \to \overline{D(A)}$.

We show that $(S_t)_{t\geq 0}$ is an integrated semigroup on $\overline{D(A)}$.

Obviously $t \to S_t$ is strongly continuous on D. Since $(S_t)_{t\geq 0}$ is by 8.2(ii) uniformly bounded on compact t-intervals (and D is dense in $\overline{D(A)}$) $t \to S_t$ is strongly continuous on $\overline{D(A)}$.

To see that $(S_t)_{t\geq 0}$ fulfills the functional equation let $x \in D$. For $s \geq 0$ the function $t \to u(t+s,x)$ is a solution for the initial value $u(s,x)$. By uniqueness $u(t+s,x) = u(t,u(s,x))$ or, in terms of the U_t's: $U_{t+s} = U_t U_s$ on D. Integration first in t and then in s yields the functional equation on D. By boundedness it extends to $\overline{D(A)}$.

(a) and (b) hold since for $x \in D$ $u(t,x) = S_t'x$.

We adress the converse question:
Is every integrated semigroup linked with a Cauchy problem, wellposed in the integrated sense?

In answering this we confine ourselves to non-degenerate integrated semigroups and we begin with a technical lemma.

Lemma 8.5:

Let $(S_t)_{t\geq 0}$ be a non-degenerate integrated semigroup. Then

(i) $S_o'':C^2 \to E$ is closable,

(ii) $D(\overline{S_o''}) \subset C^1$ and $S_o'' S_t = S_t \overline{S_o''}$ on $D(\overline{S_o''})$.

Proof:

(i): Let (x_n) be a sequence in C^2 with $x_n \to 0$ and $S_o''x_n \to y$. By prop. 2.6(i) and (ii) we have for $t \geq 0$

(+) $\quad S_t x_n = \int_o^t S_s S_o'' x_n \, ds + t x_n$.

For $n \to \infty$ we obtain $0 = \int_o^t S_s y \, ds$ for $t \geq 0$. By differenciating this and using the non-degeneracy we deduce $y = 0$.

(ii): Let $x \in D(\overline{S_o''})$. Choose a sequence (x_n) in C^2 such that $x_n \to x$ and $S_o'' x_n \to \overline{S_o''}x$. Again by (+) we derive

(++) $\quad S_t x = \int_o^t S_s \overline{S_o''} x \, ds + tx$.

The right hand side is continuously differentiable in t.

Therefore the left hand side also, which implies $x \in C^1$.

Differentiating (++) we obtain

$$S_t'x = S_t\overline{S}_o''x + x.$$

Since $D(\overline{S}_o'') \subset C^1$ we also have (prop. 2.6(i))

$$S_t'x = S_o''S_tx + x.$$

This shows $S_o''S_tx = S_t\overline{S}_o''x$.

Using lemma 8.5 we prove

Theorem 8.6:

Let $(S_t)_{t\geq 0}$ be a non-degenerate integrated semigroup.

Then for $A = \overline{S}_o''$ the Cauchy problem (*) is wellposed on C^2 in the integrated sense.

Proof:

Let $x \in C^2$. By prop. 2.6(ii) $S_t''x = S_o''S_t'x$ for $t \geq 0$ and by non-degeneracy $S_o'x = x$. This shows 8.2(i) (solvability).

8.2(ii): Let u be a solution of $u' = \overline{S}_o''u$. Consider for fixed $t \geq 0$ the function $s \to S_{t-s}u(s)$. Since $u(s) \in D(\overline{S}_o'') \subset C^1$ (lemma 8.5) this function is continuously differentiable and

$$\begin{aligned}\frac{d}{ds}(S_{t-s}u(s)) &= -S_{t-s}'u(s) + S_{t-s}u'(s)\\ &= -S_o''S_{t-s}u(s) - u(s) + S_{t-s}\overline{S}_o''u(s)\\ &\quad \text{by prop. 2.6(i)}\\ &= -u(s) \quad \text{by lemma 8.5.}\end{aligned}$$

Integration from 0 to t yields (by $S_o = 0$ and $u(0) = x$)

$$\int_o^t u(s)\,ds = S_t x.$$

This implies 8.2(ii).

A stronger statement is possible if in addition $(S_t)_{t\geq 0}$ is exponentially bounded:

Theorem 8.7

Let $A:D(A) \to E$ be an L-generator.

Then (*) is wellposed on C^2 (and in particular on $D(A^2)$) in the integrated sense

and there are constants M and ω such that

$$\left\| \int_o^t u(s)\,ds \right\| \leq Me^{\omega t}\|u(0)\|$$

for $t \geq 0$ and all solutions of $u' = Au$.

The proof is similar to the proof of theorem 8.6.

This theorem can be attributed to M. Sova who proved the assertion for densely defined operators A that fulfill condition 4.6(ii) (see [25], theorem 2.1).

PART II. APPLICATIONS

Section 1. Differential operators of higher order

The third derivative $(\frac{d}{dx})^3$ is a generator on $L^2(\dot{\mathbb{R}})$ but not on the other L^p-spaces. The same holds for the Schrödinger operator $i(\frac{d}{dx})^2$ (see [15], theorem 1.14).

But: Together with a large class of higher order differential operators $(\frac{d}{dx})^3$ and $i(\frac{d}{dx})^2$ are L-generators on these and other function spaces.

This is what we shall show in this section.

In the following we shall make extensive use of the Fourier transformation. We shall denote it by $\hat{}$ i.e. $\hat{f}(y) := (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} f(x) e^{-ixy}\, dx$. Its inverse will be denotet by $\check{}$.

We shall consider differential operators of the form $\Sigma_{j=0}^{k} a_j (\frac{d}{dx})^j$ on the function spaces

(L) $\quad C_o(\mathbb{R})$, $C_b(\mathbb{R})$, $L^p(\mathbb{R})$ for $1 \le p \le \infty$.

More precisely:

Let $a_o, \ldots, a_k \in \mathbb{C}$ and let E be one of these spaces.

Then we shall consider the operator $A:D(A) \to E$, where

$D(A) := \{ f \in E : \Sigma_{j=0}^{k} a_j D^j f \in E$ distributionally $\}$

and $Af := \Sigma_{j=0}^{k} a_j D^j f$.

A key role will be played by the polynomial $\sum_{j=0}^{k} a_j(-i)^j x^j$ (i = imaginary unit). It will be abbreviated by p(x).

In this notation the following theorem holds:

Theorem 1.1:

Let $a_0,\ldots,a_k \in \mathbb{C}$ such that

(RE) $\sup_{x\in\mathbb{R}} \mathrm{Re}(p(x)) < \infty$

Then A:D(A) → E is an L-generator on all spaces listed under (L) and the integrated semigroup is continuous with respect to the operator norm.

Remark 1.2:

1) The assumption (RE) has the following background:
In the L^2-case the Fourier transform of the integrated semigroup consists of multipliers $\int_0^t e^{p(x)s}\,ds$. (RE) now means: These multipliers are bounded.
2) It should be noted that in general A is not densely defined on the spaces $C_b(\mathbb{R})$ and $L^\infty(\mathbb{R})$.

We shall now prove theorem 1.1 in a series of four lemmas. Thereby we shall exclude the trivial case "k = 0". In other words: Throughout this section we shall assume $k \geq 1$ and $a_k \neq 0$.
We shall further use the abbreviation

$$w := \max(0, \sup_{x\in\mathbb{R}} \mathrm{Re}(p(x))).$$

Two operator families will be considered:

1) $(S_t)_{t\geq 0}$, where $S_t f = (1/\sqrt{2\pi})\ \check{\phi}_t * f$
with $\phi_t(x) := \int_0^t e^{p(x)s}\, ds$
(This will be the integrated semigroup.)

2) $(R_\lambda)_{\lambda > w}$, where $R_\lambda f := (1/\sqrt{2\pi})\ \tilde{r}_\lambda * f$
with $r_\lambda(x) := 1/(\lambda - p(x))$ (its resolvent).

First a closer look at the convolution kernels $\check{\phi}_t$ and $\tilde{r}_\lambda$.

Lemma 1.3:

For $t \geq 0$ and $\lambda > w$ the following holds:

(i) ϕ_t , $r_\lambda \in H^{1,2}$ (the first Sobolev space in $L^2(\mathbb{R})$),

(ii) $t \to \phi_t$ is continuous with respect to $\| \ \|_{1,2}$,

(iii) $\|\phi_t\|_{1,2} \leq (c + c't)e^{wt}$.

Proof:

The assumption $k \geq 1$ implies:

There is some $L_o \geq 0$ such that for $|x| \geq L_o$
$|p(x)| \geq |a_k x^k|/2$.

(i): One verifies that ϕ_t, $(\frac{d}{dx})\phi_t$, r_λ, $(\frac{d}{dx})r_\lambda$ are bounded as functions in x and further:

$|\phi_t(x)|, \ldots, |(\frac{d}{dx})r_\lambda(x)| \leq c/|x|$ (for suitable c).

This implies (i).

(ii): Let $0 \leq s \leq t$.

$$\|\phi_t - \phi_s\|_2^2 = \int_{-\infty}^{\infty} |\int_s^t e^{p(x)r}\, dr|^2\, dx$$

$$= \int_{-\infty}^{-L} |e^{p(x)t} - e^{p(x)s}|^2/|p(x)|^2\, dx$$

$$+ \int_{-L}^{L} |\int_s^t e^{p(x)r}\, dr|^2\, dx$$

$$+ \int_L^\infty |e^{p(x)t} - e^{p(x)s}|^2 / |p(x)|^2 \, dx \quad \text{for } L > L_o$$

$$\leq \int_{-\infty}^{-L} 16e^{2wt} / |a_k x^k|^2 \, dx + 2Le^{2wt} |t - s|^2$$

$$+ \int_L^\infty 16e^{2wt} / |a_k x^k|^2 \, dx .$$

If L is large the first and third term get small. By choosing $|t - s|$ small one can make the second term small, too.

$$\| (\tfrac{d}{dx}) \phi_t - (\tfrac{d}{dx}) \phi_s \|_2^2$$

$$\leq ct^2 e^{2wt} \int_{-\infty}^{-L} |x|^{-2} \, dx$$

$$+ 2Lt^2 e^{2wt} (\sup_{|x| \leq L} |p'(x)|)^2 |t - s|^2$$

$$+ ct^2 e^{2wt} \int_L^\infty |x|^{-2} \, dx$$

(in a similar calculation).

Choose again L large then $|t - s|$ small.

(iii) can be obtained by choosing $L = L_o$ and $s = 0$.

Lemma 1.4:

Let $f \in H^{1,2}$.

Then $\hat{f} \in L^1(\mathbb{R})$ and

there is a constant $c > 0$ such that

$$\|\hat{f}\|_1 \leq c \|f\|_{1,2} \quad \text{for } f \in H^{1,2}.$$

Proof:

Let $f \in H^{1,2}$. Then $(f + (\tfrac{d}{dx}) f)^\wedge \in L^2$.

This implies: The function $y \to (1 + iy)\hat{f}(y)$ is in L^2.

Therefore

$$\left(\int_{-\infty}^{\infty} |\check{f}(y)|\,dy\right)^2 = \left(\int_{-\infty}^{\infty} \frac{1}{|1+iy|}\,|(1+iy)\check{f}|\,dy\right)^2$$

$$\leq \int_{-\infty}^{\infty} |1+iy|^{-2}\,dy \int_{-\infty}^{\infty} |(f + (\tfrac{d}{dx})f)|^2\,dy\,.$$

(Cauchy-Schwarz)

This shows that $\check{f} \in L^1(\mathbb{R})$ and that the estimate holds with $c := \left(\int_{-\infty}^{\infty} |1+iy|^{-2}\,dy\right)^{1/2}$.

Lemma 1.5:

Let E be one of the spaces listed under (L).

Then the family $(S_t)_{t\geq 0}$, $S_t f = (1/\sqrt{2\pi})\ \check{\phi}_t * f$, is an integrated semigroup on E continuous with respect to the operator norm.

Further:

(i) $\|S_t\|_{L(E)} \leq (a+bt)e^{wt}$ (with suitable constants $a,b \geq 0$).

(ii) For $\lambda > w$ the pseudo-resolvent of $(S_t)_{t\geq 0}$ is given by $R_\lambda f = (1/\sqrt{2\pi})\ \check{r}_\lambda * f$.

Proof:

The lemmas 1.3 and 1.4 imply:

1) $\check{\phi}_t \in L^1$ for $t \geq 0$

2) $t \to \check{\phi}_t$ is continuous with respect to $\|\ \|_1$.

3) $\|\check{\phi}_t\|_1 \leq (a+bt)e^{wt}$ for suitable $a,b \geq 0$.

By 1) the convolution operators S_t are bounded operator on all listed spaces.

By the Young-inequality

$$\|g * f\|_p \leq \|g\|_1\ \|f\|_p \quad \text{for } g \in L^1,\ f \in L^p,\ 1 \leq p \leq \infty$$

2) implies the norm continuity and 3) implies (i).

To show that $(S_t)_{t\geq 0}$ fulfills the functional equation observe that

$$\phi_s \, \phi_t = \int_0^s (\phi_{r+t} - \phi_r) \, dr \, .$$

Therefore

$$(1/\sqrt{2\pi}) \, \hat{\phi}_s * \hat{\phi}_t = \int_0^s (\hat{\phi}_{r+t} - \hat{\phi}_r) \, dr.$$

And since the convolution is associative the functional equation follows.

(ii): By (i) $\|\hat{\phi}_t\| \leq Me^{\omega t}$ (and therefore $\|S_t\| \leq Me^{\omega t}$) for any $\omega > w$ (with suitably chosen M). Thus the Laplace integral exists for $\lambda > w$.

Let $\lambda > w$. We show first (by means of distributions)

$$\lambda \int_0^\infty e^{-\lambda t} \, \hat{\phi}_t \, dt = \hat{r}_\lambda .$$

Let ψ be a function of rapid decrease. Then

$$\begin{aligned}
&\int_{-\infty}^{\infty} \lambda \int_0^\infty e^{-\lambda t} \, \hat{\phi}_t(x) \, dt \, \psi(x) \, dx \\
&= \int_0^\infty \int_{-\infty}^{\infty} \lambda e^{-\lambda t} \, \hat{\phi}_t(x) \, \psi(x) \, dx \, dt \\
&= \int_0^\infty \int_{-\infty}^{\infty} \lambda e^{-\lambda t} \, \phi_t(x) \, \hat{\psi}(x) \, dx \, dt \\
&= \int_{-\infty}^{\infty} \lambda \int_0^\infty e^{-\lambda t} \, \phi_t(x) \, dt \, \hat{\psi}(x) \, dx \\
&= \int_{-\infty}^{\infty} r_\lambda(x) \, \hat{\psi}(x) \, dx \\
&= \int_{-\infty}^{\infty} \hat{r}_\lambda(x) \, \psi(x) \, dx \, .
\end{aligned}$$

This shows the assertion.

Now let for the moment $f \in L^\infty(\mathbb{R})$. Then for $z \in \mathbb{R}$ we have

$$\begin{aligned}
&\lambda \int_0^\infty e^{-\lambda t} S_t \, f \, dt \, (z) \\
&= \lambda \int_0^\infty e^{-\lambda t} \, (1/\sqrt{2\pi}) \, (\hat{\phi}_t * f)(z) \, dt \\
&= (1/\sqrt{2\pi}) \, \lambda \int_0^\infty \int_{-\infty}^{\infty} e^{-\lambda t} \hat{\phi}_t(z - y) f(y) \, dy \, dt \, .
\end{aligned}$$

The integrand is in $L^1(\mathbb{R}_+ \times \mathbb{R})$. Therefore one obtains by Fubini's theorem

$$= (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} \lambda \int_0^\infty e^{-\lambda t} \hat{\phi}_t(z - y) \, dt \, f(y) \, dy.$$

And from what we just showed we deduce

$$= (1/\sqrt{2\pi})\, \tilde{r}_\lambda * f(z) = R_\lambda f(z).$$

From this (ii) follows for L^∞, C_b and C_o. Since the functions of rapid decrease are in L^∞ and dense in all other L^p-spaces the rest follows by the boundedness of R_λ (this is obtained from lemma 1.3 (i) in combination with lemma 1.4).

The proof of theorem 1.1 is completed by

Lemma 1.6:

A:D(A) $\to$ E is the L-generator of $(S_t)_{t\geq 0}$ for all spaces E listed under (L).

Proof:

The assertion is shown by distribution techniques. Let $f \in E$ and Ψ be a function of rapid decrease.

In the following Ψ_- is the function, defined by $\Psi_-(x) := \Psi(-x)$.

Then we have

$$\begin{aligned}
&\langle(\lambda - \Sigma_{j=o}^k a_j D^j) R_\lambda f, \Psi\rangle \\
&= \int_{-\infty}^{\infty} (1/\sqrt{2\pi})(\tilde{r}_\lambda * f)(x)\ (\lambda\Psi(x) - \Sigma_{j=o}^k a_j (-1)^j \Psi^{(j)}(x))\, dx \\
&= \int_{-\infty}^{\infty} (1/\sqrt{2\pi})(\tilde{r}_\lambda * f)(x)\ (\lambda\Psi_- - \Sigma_{j=o}^k a_j \Psi_-^{(j)})(-x)\, dx
\end{aligned}$$

or shorter

$$= (1/\sqrt{2\pi})(\tilde{r}_\lambda * f) * (\lambda\Psi_- - \Sigma_{j=o}^k a_j \Psi_-^{(j)})(0)$$

$$= f * ((1/\sqrt{2\pi})\tilde{r}_\lambda * (\lambda\Psi_- - \Sigma_{j=o}^k a_j \Psi_-^{(j)}))(0)$$

$= f * (r_\lambda (\lambda - p(\cdot))\ (\Psi_-)\hat{}\ )\check{}\ (0)$

$= f * \Psi_- (0) = \int_{-\infty}^{\infty} f(x)\ \Psi(x)\ dx$.

This implies the distributional equation

$$\Sigma_{j=0}^{k} a_j D^j R_\lambda f = (\lambda R_\lambda f - f).$$

The distribution on the right hand side is in E.

Therefore $R_\lambda f \in D(A)$ and $(\lambda - A)R_\lambda f = f$.

In a similar argument one shows

$$R_\lambda(\lambda - A)f = f \quad \text{for} \quad f \in D(A).$$

This completes the proof.

Remark 1.7:

The natural question that arises here is:

Can theorem 1.1 be generalized for the $\mathbb{R}^n$-case?

If we try to use the preceding proof for $\mathbb{R}^n$ an obstacle comes up: Polynomials in $\mathbb{R}^n$ need not grow as fast as it is needed in the proof.

And even if we require strong enough growth of p we only get a weaker result:

Consider $A = \Sigma_{|\alpha| \le k} a_\alpha D^\alpha$, where

$\alpha := (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| := \Sigma_{j=1}^{n} \alpha_j$ and

$D^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \ldots (\frac{\partial}{\partial x_n})^{\alpha_n}$.

Define $p(x) := \Sigma_{|\alpha| \le k} a_\alpha (-i)^{|\alpha|} x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and

$\operatorname{ord}(p) := \max \{ |\alpha| : a_\alpha \neq 0 \}$.

$\Sigma_{|\alpha| = \operatorname{ord}(p)} a_\alpha (-i)^{|\alpha|} x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ is called the principal part of p(x).

The growth condition on p that we need is equivalent to the socalled "ellipticity":

A polynomial $p(x)$ is called <u>elliptic</u> if its principal part vanishes only in $x = 0$.

<u>Lemma 1.8:</u>

Equivalent are:

(i) $p(x)$ is elliptic.

(ii) There are constants L_o, $c > 0$ such that

$$|p(x)| \geq c|x|^{ord(p)} \quad \text{for } |x| \geq L_o.$$

Proof:

We denote the principal part of $p(x)$ by $q(x)$ and abbreviate $m := ord(p)$.

The key property is $q(\lambda x) = \lambda^m q(x)$ for $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

=>: By continuity and compactness reasons we have $|q(y)| \geq c > 0$ for $|y| = 1$. This implies by the above property for arbitrary $x \in \mathbb{R}^n$

$$c \leq |q(\frac{x}{|x|})| = (\frac{1}{|x|})^m |q(x)|.$$

Therefore

$$(\frac{1}{|x|})^m |p(x)| = (\frac{1}{|x|})^m |q(x)| - (\frac{1}{|x|})^m |p(x) - q(x)|$$
$$\geq c - \frac{c}{2} \quad \text{for } |x| \geq L_o$$

if L_o is chosen large enough (since $ord(p-q) < m$).

<=: Let (ii) hold and $q(x) = 0$ for some $x \in \mathbb{R}^n$.

Then $q(\lambda x) = \lambda^m q(x) = 0$ for arbitrary $\lambda > 0$. This implies $c|\lambda x|^m \leq |p(\lambda x)| = |p(\lambda x) - q(\lambda x)|$. Therefore $c|x|^m \leq |p(\lambda x) - q(\lambda x)|/\lambda^m$. Since $ord(p-q) < m$ the right hand term goes to zero as $\lambda \to \infty$. Thus $x = 0$.

The following theorem replaces theorem 1.1 in the $\mathbb{R}^n$-case.

Theorem 1.9:

Let $p(x) := \sum_{|\alpha| \le k} a_\alpha (-i)^{|\alpha|} x_1^{\alpha_1} \dots x_n^{\alpha_n}$ be an elliptic polynomial with $\operatorname{ord}(p) > \frac{n}{2}$ and

(RE) $\sup_{x \in \mathbb{R}^n} \operatorname{Re} p(x) < \infty$.

Then A is the L-generator of an $[\frac{n}{2}]+2$ - times integrated semigroup on the spaces

(L^n) $C_o(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$.

Remark 1.10:

Elliptic polynomials that fulfill (RE) are necessarily of even order (if $n \ge 2$).

We only sketch the proof:

Define $m := [\frac{n}{2}]+2$. We have to show:

$$S_t f := (\tfrac{1}{2\pi})^{n/2} \check{\phi}_t * f \text{ , where}$$

$$\phi_t := \frac{1}{(m-1)!} \int_o^t (t-s)^{m-1} e^{p(x)s} \, ds$$

$$= e^{p(x)t}/p(x)^m - \sum_{j=o}^{m-1} \frac{1}{j!} t^j/p(x)^{m-j}$$

defines an integrated semigroup with pseudo-resolvent $(R_\lambda)_{\operatorname{Re}\lambda > \omega}$, where $R_\lambda f := (\frac{1}{2\pi})^{n/2} \check{r}_\lambda * f$ and

$$r_\lambda(x) := \frac{1}{\lambda - p(x)} .$$

Further $R_\lambda := (\lambda - A)^{-1}$.

To this purpose one verifies by induction that (for any polynomial $p(x)$) the following hold:

1) $D^{\alpha}(e^{p(x)t}) = q(x)e^{p(x)t}/p(x)^{m+|\alpha|}$,

where $q(x)$ is a sum of products $p_1(x)\dots p_s(x)$, $s \leq 2|\alpha|$ and each $p_j(x)$ is of the form $D^{\nu}p(x)$,

2) $D^{\alpha}(1/p(x)^r) = q(x)/p(x)^{r+|\alpha|}$,

where $q(x)$ is a sum of products $p_1(x)\dots p_s(x)$, $s \leq |\alpha|$ and each $p_j(x)$ is of the form $D^{\nu}p(x)$.

Using this one shows

Lemma 1.11:

Let $p(x)$ fulfill the assumptions of theorem 1.9.

Then there are constants $L_o, c > 0$ such that

$|D^{\alpha}\phi_t(x)|$, $|D^{\alpha}r_{\lambda}(x)| \leq c/|x|^{ord(p)}$

for $|x| \geq L_o$ and $|\alpha| \leq [\frac{n}{2}] + 1$.

Lemma 1.11 implies

Lemma 1.12:

Let $p(x)$ fulfill the assumptions of theorem 1.9.

Then for $t \geq 0$ and $\lambda > \omega$ the following hold

(i) $\phi_t, r_{\lambda} \in H^{m-1,2}$,

(ii) $t \to \phi_t$ is continuous with respect to $\|\ \|_{H^{m-1,2}}$,

(iii) $\|\phi_t\|_{H^{m-1,2}} \leq (c + c't^m)e^{wt}$.

The proof of theorem 1.1 can be copied now if we replace lemma 1.4 by

Lemma 1.13:

Let $k > \frac{n}{2}$.

Then the inverse Fourier transform maps $H^{k,2}(\mathbb{R}^n)$ boundedly into $L^1(\mathbb{R}^n)$ (i.e. $\|\check{f}\|_L 1 \leq c\|f\|_H k,2$).

The proof is similar to the proof of lemma 1.4 and therefore is omitted.

A consequence of theorem 1.9 is:
For $n \leq 3$ the Schrödinger operator $i\Delta$ is the L-generator of a three-times integrated semigroup on all spaces listed under (L^n).

$i\Delta$ and other elliptic operators were studied on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, also within the framework of distribution semigroups.
S. Apteker proved ([2], p. 84):
If $p(x)$ is an elliptic polynomial and (RE) holds then A generates a distribution semigroup on $L^p(\mathbb{R}^n)$ for $1<p<\infty$.
For refinements of this and other results see also [6] and [7].

Section 2. Cauchy problems of second order

We shall consider Cauchy problems of the form

$$(*)\qquad \begin{aligned} u''(t) &= A_1 u'(t) + A_0 u(t) \\ u(0) &= x \\ u'(0) &= y, \end{aligned}$$

where $A_i : D(A_i) \to E$ are densely defined operators.

Of particular importance are the Cauchy problems

$$(**)\qquad \begin{aligned} u''(t) &= Au(t) \\ u(0) &= x \\ u'(0) &= y. \end{aligned}$$

We start with these.

For wellposedness we require solvability and uniqueness for $x,y \in D(A)$. (A solution is a C^2-function $u:\mathbb{R}_+ \to E$ such that $u(t) \in D(A)$ for all $t \geq 0$ and (**) holds.) If the solution for the initial values x and y is written as $u(*;x,y)$ then the following continuity is required: For any compact t-interval I there is a constant M_I such that

$$\sup_{t \in I} \|u(t,x,y)\| \leq M_I(\|x\| + \|y\|) \quad \text{for all } x,y \in D(A).$$

For these Cauchy problems the theory of cosine functions was developed. We refer for the terminology (e.g. generator, Sova's conditions) to [12]. The following theorem shows that the theory of integrated semigroups allows a look at cosine functions from a new perspective.

Theorem 2.1:

Let $A:D(A) \to E$ be a linear operator.

Equivalent are:

(i) A is generator of a cosine function.

(ii) $\mathcal{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ is an L-generator on the Banach space $E \times E$.

Proof:

(i) => (ii):

Let $(Cos(t))_{t \in \mathbb{R}}$ be the cosine function for A.

It follows from the theory of cosine functions([12]):

a) $\|Cos(t)\| \leq Me^{\omega t}$ for $t \geq 0$ (M, ω suitable).

b) For $\lambda > \omega$ $\lambda(\lambda^2 - A)^{-1} = \int_0^\infty e^{-\lambda t} Cos(t)\, dt$.

Let $Sin(t) := \int_0^t Cos(s)\, ds$.

Then for the family $(S_t)_{t \geq 0}$, where

$$S_t := \begin{pmatrix} Sin(t) & \int_0^t Sin(s)\, ds \\ Cos(t) - I & Sin(t) \end{pmatrix}$$

the following holds:

$\|S_t\| \leq M'e^{\omega' t}$ for $t \geq 0$ (M', ω' suitable).

After a short calculation one obtains

$$\lambda \int_0^\infty e^{-\lambda t} S_t\, dt = \begin{pmatrix} \lambda & 1 \\ A & \lambda \end{pmatrix} (\lambda^2 - A)^{-1} \quad \text{for } \lambda > \omega'.$$

The term on the right hand side is just $(\lambda - \mathcal{A})^{-1}$.

Therefore according to part I, remark 2.3 $(S_t)_{t \geq 0}$ is an integrated semigroup and $\mathcal{A}$ its L-generator.

(ii) => (i):

Let $(S_t)_{t \geq 0}$ be the integrated semigroup for the L-genera-

tor A. We try to fulfill Sova's conditions ([12],[23]).

For suitable M,ω $\quad \|S_t\| \leq Me^{\omega t}$.

For $\mathrm{Re}\lambda > \omega$ one obtains

$$(\tfrac{d}{d\lambda})^n(\tfrac{1}{\lambda}(\lambda - A)^{-1}) = (\tfrac{d}{d\lambda})^n \int_o^\infty e^{-\lambda t}S_t\, dt .$$

Thus $\quad \|(\tfrac{d}{d\lambda})^n(\tfrac{1}{\lambda}(\lambda - A)^{-1})\| \leq n!\int_o^\infty t^n e^{-\mathrm{Re}\lambda t} Me^{\omega t}\, dt$

$$= Mn!/(\mathrm{Re}\lambda - \omega)^{n+1} .$$

For $\mathrm{Re}\lambda > \omega$ we have the matrix equations

$$(\lambda - A)(\lambda - A)^{-1} = I \quad \text{and}$$

$$(\lambda - A)^{-1}(\lambda - A) = I_{|D(A)}.$$

Inspecting these equations in detail one deduces that $\lambda^2 - A$ is invertible and

$$(\lambda - A)^{-1} = \begin{pmatrix} \lambda & 1 \\ A & \lambda \end{pmatrix} (\lambda^2 - A)^{-1} .$$

The above estimate for $\|(\tfrac{d}{d\lambda})^n(\tfrac{1}{\lambda}(\lambda - A)^{-1})\|$ is then valid for all entries, in particular

$$\|(\tfrac{d}{d\lambda})^n(\tfrac{A}{\lambda}(\lambda^2 - A)^{-1})\| \leq Mn!/(\mathrm{Re}\lambda - \omega)^{n+1} .$$

Since $\frac{A}{\lambda}(\lambda^2 - A)^{-1} = \lambda(\lambda^2 - A)^{-1} - \frac{1}{\lambda}$ we obtain from this (assuming without loss of generality $\omega \geq 0$):

$$\|(\tfrac{d}{d\lambda})^n(\lambda(\lambda^2 - A)^{-1})\|$$

$$\leq n!\,(\,1/|\lambda|^{n+1} + M/(\mathrm{Re}\lambda - \omega)^{n+1}$$

$$\leq n!(M + 1)/(\mathrm{Re}\lambda - \omega)^{n+1}.$$

This is one part of Sova's conditions. The other one is the density of D(A):

$S_t = \begin{pmatrix} T_t & U_t \\ V_t & W_t \end{pmatrix}$ for suitable operators $T_t, U_t, V_t, W_t \in L(E)$.

Thus the equation $(\lambda - A)^{-1} = \lambda\int_o^\infty e^{-\lambda t}S_t\, dt$ implies

$$\lambda^2(\lambda^2 - A)^{-1} - I = A(\lambda^2 - A)^{-1} = \lambda\int_o^\infty e^{-\lambda t}V_t dt .$$

For $x \in E$ we obtain

$$\begin{aligned} & \|\lambda^2(\lambda^2 - A)^{-1}x - x\| \\ \leq\ & \lambda\int_0^\infty e^{-\lambda t}\|V_t x\|dt \\ =\ & \lambda\int_0^\delta e^{-\lambda t}\|V_t x\|dt + \lambda\int_\delta^\infty e^{-\lambda t}\|V_t x\|dt \\ \leq\ & \sup_{0\leq t\leq\delta}\|V_t x\| + \frac{M\lambda}{\lambda-\omega}e^{-(\lambda-\omega)\delta}\|x\| . \end{aligned}$$

For small δ the first term becomes small (since $V_t x \to 0$ as $t \to 0$). Let finally $\lambda \to \infty$ to make the second term small,too.

Theorem 2.1 yields the tool to prove existence and perturbation theorems by the methods of integrated semigroup theory. The following theorem illustrates this.

Theorem 2.2 (bounded perturbations):

Let A be generator of a cosine function

and $B \in L(E)$.

Then $A + B$ generates a cosine function.

Proof:

By theorem 2.1 $\mathcal{A} := \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}$ is an L-generator on $E \times E$.

$\mathcal{B} := \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$ is a bounded operator on $E \times E$ and

$$\mathcal{B}\mathcal{A} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} .$$

Therefore $\|\mathcal{B}\mathcal{A}(x,y)\|_{E\times E} \leq \|B\|_{L(E)}\|y\|_E$

$\leq \|B\|_{L(E)}\|(x,y)\|_{E\times E}$ for $(x,y) \in D(\mathcal{A})$.

By theorem 6.7 one obtains that $\mathcal{A} + \mathcal{B} = \begin{pmatrix} 0 & I \\ A + B & 0 \end{pmatrix}$ is

is an L-generator. The assertion follows by theorem 2.1.

Remark 2.3:

We go back to the diagrams in part I, section 3.
Consider a cosine function with generator A and let $\hat{A}$ be the matrix as above.
With $\mathcal{D}^n := \{ x \in E : t \to \mathrm{Cos}(t)x$ is a C^n-function $\}$
the first diagram takes the form

$$\begin{array}{ccc} & & E \times E \\ C^1 = \mathcal{D}^1 \times E & & \\ & & D(\hat{A}) = D(A) \times E \\ C^2 = \mathcal{D}^2 \times \mathcal{D}^3 & & \\ & & D(\hat{A}^2) = D(A) \times D(A) \\ C^3 = \mathcal{D}^3 \times \mathcal{D}^4 & & \\ & & D(\hat{A}^3) = D(A^2) \times D(A) \end{array}$$

Thereby we dropped the arrows and the statements about density since we shall concentrate on the inclusions.
A meditation about these product spaces yields the following diagram for the spaces $\mathcal{D}^n$ and $D(A^n)$:

$$\begin{array}{ccc} & & E \\ \mathcal{D}^1 & & \\ \mathcal{D}^2 & = & D(A) \\ \mathcal{D}^3 & & \\ \mathcal{D}^4 & = & D(A^2) \\ & \text{etc.} & \end{array}$$

This shows that D(A) is just the second "differentiation space" of the cosine function. Usually this is the way the domain of the generator is defined. In general the

inclusions in the $\mathcal{D}^n$-chain are proper. This means for the C^n-$D(A^n)$-diagram: All inclusions are proper.

Instead of doing further investigations about the connection between cosine functions and integrated semigroups we shall address a problem that has not been accessible to an operator theoretical treatment till now (to the best of my knowledge):

The "complete" Cauchy problem or second order

$$(*)\qquad \begin{aligned} u''(t) &= A_0u(t) + A_1u'(t) \\ u(0) &= x \\ u'(0) &= y\ . \end{aligned}$$

We formulate first the kind of wellposedness that we will use. Thereby we use the same terminology as in section 8 of part I.

Definition 2.4:

Let $A_i:D(A_i) \to E$ $(i = 0,1)$ be linear densely defined operators.

A solution of $u'' = A_0u + A_1u'$ is a C^2-function $u:[0,\infty) \to E$ such that $u(t) \in D(A_0)$, $u'(t) \in D(A_1)$ and $u''(t) = A_0u(t) + A_1u'(t)$ holds for $t \geq 0$.

If $x \in D(A_0)$ and $y \in D(A_1)$ then (*) is solvable for (x,y) if there is a solution of $u'' = A_0u + A_1u'$ with $u(0) = x$ and $u'(0) = y$.

Let D be a subspace of $D(A_o) \times D(A_1)$, dense in $E \times E$.

(*) is called <u>wellposed on D</u> if

(i) (*) is solvable for all $(x,y) \in D$,

(ii) for every compact t-interval I there is a $K_I \geq 0$ such that:

$$\sup_{t \in I} \|u(t)\| \leq K_I(\|u(0)\| + \|u'(0)\|)$$

for all solutions of $u'' = A_o u + A_1 u'$.

<u>Remark 2.5:</u>

In the following we shall again make use of a 2×2-matrix namely

$$\begin{pmatrix} 0 & 1 \\ A_o & A_1 \end{pmatrix} : D(A_o) \times D(A_1) \to E \times E.$$

It will be denoted by $A: D(A) \to E \times E$.

To deal with this wellposedness within the framework of integrated semigroups we reformulate it in the sense of part I, section 8:

<u>Theorem 2.6:</u>

Let $A_i: D(A_i) \to E$ $(i = 0,1)$ be linear densely defined operators and D a subspace of $D(A_o) \times D(A_1)$, dense in $E \times E$.

Equivalent are:

(i) (*) is wellposed on D.

(ii) The Cauchy problem $v'(t) = Av(t)$, $v(0) = \begin{pmatrix} x \\ y \end{pmatrix}$ is wellposed in the integrated sense on D.

The proof is simple and can be omitted.

Remark 2.7:

We use theorem 2.6 to answer the question:

How is this kind of wellposedness in the case "$A_1 = 0$" connected with the wellposedness formulated for (**)?

In fact:

If for a closed operator $A:D(A) \to E$ (**) is wellposed then for $A_o = A$, $A_1 = 0$ (*) is wellposed on $D(A)\times D(A)$.

This holds because:

In this case A is the generator of a cosine function $(Cos(t))_{t\in\mathbb{R}}$ (Fattorini [12] p. 28) or: $A:D(A) \to E \times E$ is the L-generator of the integrated semigroup $(S_t)_{t\geq 0}$,

$$S_t := \begin{pmatrix} Sin(t) & \int_o^t Sin(s)\, ds \\ Cos(t) - I & Sin(t) \end{pmatrix}$$

(proof of theorem 2.1).

By part I, theorem 8.7 the Cauchy problem $v'(t) = Av(t)$, $v(0) = \begin{pmatrix} x \\ y \end{pmatrix}$ is wellposed in the integrated sense on $D(A^2) = D(A)\times D(A)$. By theorem 2.6 (*) is wellposed on $D(A)\times D(A)$.

A new phenomenon arises in connection with the complete Cauchy problem of second order:

The solutions may grow extremely fast.

Thereby the growth function is defined as

$\omega:\mathbb{R}_+ \to \mathbb{R}_+$, $\omega_t := \sup_{\|x\|+\|y\| \leq 1,\ 0\leq s\leq t} \|u(s;x,y)\|$

(where $(x,y) \in D$ and $u(*;x,y)$ is the solution with initial values $u(0) = x$ and $u'(0) = y$).

Fattorini ([12], p. 273) showed that any L^∞_{loc}-function on $[1,\infty)$ can be topped by a growth function, i.e. for any function $f:[1,\infty) \to \mathbb{R}_+$, bounded on compact t-intervals there are operators $A_i:D(A_i) \to E$ $(i = 0,1)$ and a set D such that (*) is wellposed on D and the growth function ω fulfills $\omega_t \geq f_t$ for $t \geq 1$.

Let $(S_t)_{t\geq 0}$ the integrated semigroup connected with the Cauchy problem $v'(t) = \mathcal{A}v(t)$, $v(0) = \begin{matrix} x \\ y \end{matrix}$ according to theorem 8.4 of part I. If $(S_t)_{t\geq 0}$ is written as

$S_t = \begin{pmatrix} U_t & V_t \\ P_t & Q_t \end{pmatrix}$ then a short calculation shows that

$P_t x + Q_t y + x = u(t;x,y)$ for solutions $u(*;x,y)$.

The inequality $\omega_t \geq f_t$ implies now that

$$\sup_{0\leq s\leq t} \|S_s\| \geq \omega_t - 1 \geq f_t - 1 \quad \text{for } t \geq 1.$$

This means in particular that integrated semigroups may grow arbitrarily fast.

This also implies that if $(S_t)_{t\geq 0}$ is exponentially bounded then the growth function is exponentially bounded, too.

In the following we shall restrict ourselves to exponentially bounded growth functions and we ask:

What operators A_o and A_1 make the Cauchy problem (*) wellposed and yield an exponentially bounded growth function?

A sufficient condition is certainly given by

"The operator $\mathcal{A}$ (of remark 2.5) is an L-generator."

and in the following we shall try to specify this condition. First a technical lemma:

Lemma 2.8:

Let $A_i : D(A_i) \to E$ $(i = 0,1)$ be linear densely defined operators such that also $D(A_0) \cap D(A_1)$ is dense in E and let $\lambda \in \mathbb{C}$, $\lambda \neq 0$.

With the operator $A : D(A) \to E \times E$ the following assertions are equivalent:

(i) $\lambda - A$ is invertible.

(ii) a) The operator

$$\Delta_\lambda := (\lambda(\lambda - A_1) - A_0)\,|_{D(A_0)\cap D(A_1)}$$

is invertible.

b) $\Delta_\lambda^{-1}A_0$ has a bounded extension $\overline{\Delta_\lambda^{-1}A_0}$ and $\overline{\Delta_\lambda^{-1}A_0}\, E \subset D(A_1)$.

c) $\Delta_\lambda^{-1}A_1$ has a bounded extension $\overline{\Delta_\lambda^{-1}A_1}$ and $\overline{\Delta_\lambda^{-1}A_1}\, E \subset D(A_0)$.

d) $(\lambda - A_1)\overline{\Delta_\lambda^{-1}A_0} = A_0\overline{\Delta_\lambda^{-1}}(\lambda - A_1)$.

If A_0 and A_1 are closed (i) is equivalent to

(iii) a) Δ_λ is invertible.

b') One of the operators $\Delta_\lambda^{-1}A_0$ and $\Delta_\lambda^{-1}A_1$ has a bounded extension.

c') One of the operators $A_0\Delta_\lambda^{-1}A_1$ and $A_1\Delta_\lambda^{-1}A_0$ has a bounded extension.

Proof:

We do not go into the details because the proof purely technical.

(ii) => (i): The inverse of $\lambda - A$ is given by

$$\begin{pmatrix} \overline{\Delta_\lambda^{-1}(\lambda - A_1)} & \Delta_\lambda^{-1} \\ \overline{\Delta_\lambda^{-1}A_0} & \lambda\Delta_\lambda^{-1} \end{pmatrix},$$

as one can show in a short calculation.

(i) => (ii): Evaluating all equations that are contained in the matrix equations

$$(\lambda - A)(\lambda - A)^{-1} = I$$

$$(\lambda - A)^{-1}(\lambda - A) = I_{|D(A)}$$

one obtains (ii) a) - d).

(ii) => (iii): The alternative in b') is based on the equation

$$\begin{aligned} I_{|D(A)\cap D(A)} &= \Delta_\lambda^{-1}\Delta_\lambda \\ &= \lambda^2\Delta_\lambda^{-1} - \lambda\Delta_\lambda^{-1}A_1 - \Delta_\lambda^{-1}A_0 . \end{aligned}$$

The last term is bounded if and only if the next to the last term is bounded.

c') is based on the identity

$$(\lambda - A_1)\Delta_\lambda^{-1}A_0 = A_0\Delta_\lambda^{-1}(\lambda - A_1) \quad \text{on } D(A_0)\cap D(A_1)$$

or $\lambda\Delta_\lambda^{-1}A_0 - \lambda A_0\Delta_\lambda^{-1} = A_1\Delta_\lambda^{-1}A_0 - A_0\Delta_\lambda^{-1}A_1$.

The left hand side is bounded by b') and the closedness of A_0. Therefore the alternative in c') holds.

(ii) => (iii) can be shown without major problems.

Together with the theorems 4.4 and 4.6 of part I this theorem yields a characterization of L-generators A with the property " $D(A_0)\cap D(A_1)$ is dense in E ":

Theorem 2.9:

Let $A_i : D(A_i) \to E$ $(i = 0,1)$ be linear operators and $D(A_0) \cap D(A_1)$ dense in E.

Equivalent are:

(i) $\mathcal{A}$ is an L-generator.

(ii) There are constants $M, \omega \geq 0$ with the properties:

For $\lambda > \omega$ 3.8(ii) holds and the following inequalities:

$$\left\| \left(\frac{d}{d\lambda}\right)^n \Delta_\lambda^{-1} \right\| \leq n!M/(\lambda - \omega)^{n+1}$$

$$\left\| \left(\frac{d}{d\lambda}\right)^n (\Delta_\lambda^{-1}(\lambda - A_1))x \right\| \leq n!M/(\lambda - \omega)^{n+1} \|x\|$$

for $n \in \mathbb{N}$ and $x \in D(A_1)$.

M. Sova proved a version of this on the level of Cauchy problems in [24] (even without the assumption " $D(A_0) \cap D(A_1)$ is dense in E ").

Proof:

(i) => (ii): By part I, theorem 4.4 there are $M, \omega \geq 0$ such that $(\omega, \infty) \subset \rho(A)$ and

(+) $\left\| \left(\frac{d}{d\lambda}\right)^n \left(\frac{1}{\lambda}(\lambda - A)^{-1}\right) \right\| \leq n!M/(\lambda - \omega)^{n+1}$.

This implies 2.8(ii) for $\lambda > \omega$.

In the proof of lemma 2.8 we mentioned that

$$(\lambda - A)^{-1} = \begin{pmatrix} \overline{\Delta_\lambda^{-1}(\lambda - A_1)} & \Delta_\lambda^{-1} \\ \\ \overline{\Delta_\lambda^{-1} A_0} & \lambda \Delta_\lambda^{-1} \end{pmatrix}.$$

Since (+) is valid for all entries we immediately obtain

the first estimate.

Since also $\|(\frac{d}{d\lambda})^n(\overline{\frac{1}{\lambda}\Delta_\lambda^{-1}A_o})\| \leq n!M/(\lambda-\omega)^{n+1}$ follows we deduce by $\overline{\Delta_\lambda^{-1}(\lambda - A_1)} = \frac{1}{\lambda} + \overline{\frac{1}{\lambda}\Delta_\lambda^{-1}A_o}$ that

$$\begin{aligned}\|(\tfrac{d}{d\lambda})^n(\overline{\Delta_\lambda^{-1}(\lambda - A_1)})\| &\leq |(\tfrac{d}{d\lambda})^n(\tfrac{1}{\lambda})| + \|(\tfrac{d}{d\lambda})^n(\overline{\tfrac{1}{\lambda}\Delta_\lambda^{-1}A_o})\| \\ &\leq n!/\lambda^{n+1} + n!M/(\lambda-\omega)^{n+1} \\ &\leq n!(M+1)/(\lambda-\omega)^{n+1} \text{ for } \omega \geq 0.\end{aligned}$$

(ii) => (i): By lemma 2.8 $\lambda - A$ is invertible for $\lambda > \omega$.

Consider $(\lambda - A)^{-1}$ as written in the above form.

It is not difficult to see that

$$\overline{(\tfrac{d}{d\lambda})^n(\Delta_\lambda^{-1}(\lambda - A_1))} = (\tfrac{d}{d\lambda})^n(\overline{\Delta_\lambda^{-1}(\lambda - A_1)}).$$

We have $\frac{1}{\lambda}\Delta_\lambda^{-1}A_o = \Delta_\lambda^{-1}(\lambda - A_1) - \frac{1}{\lambda}$ on $D(A_o)\cap D(A_1)$.

This implies $\overline{\frac{1}{\lambda}\Delta_\lambda^{-1}A_o} = \overline{\Delta_\lambda^{-1}(\lambda - A_1)} - \frac{1}{\lambda}$.

Therefore we obtain as above

$$\|(\tfrac{d}{d\lambda})^n(\overline{\Delta_\lambda^{-1}A_o})\| \leq n!(M+1)/(\lambda-\omega)^{n+1}.$$

This shows that the lower two entries fulfill the estimates of part I, theorem 4.6. We have to show: The other two entries fulfill these estimates, too.

This is done by the subsequent lemma.

Lemma 2.10:

Let $M, \omega \geq 0$ and $(B_\lambda)_{\lambda>\omega}$ a family of operators $B_\lambda \in L(E)$, infinitely often differentiable in λ.

Suppose further that

$$\|(\tfrac{d}{d\lambda})^n B_\lambda\| \leq n!M/(\lambda-\omega)^{n+1} \quad \text{for } \lambda > \omega,\ n \in \mathbb{N}.$$

Then

$$\|(\tfrac{d}{d\lambda})^n(\tfrac{1}{\lambda}B_\lambda)\| \leq n!M/(\lambda-\omega-1)^{n+1}$$

for $\lambda > \omega + 1$ and $n \in \mathbb{N}$.

Proof:

Let $\lambda > \omega + 1$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}\left\| \left(\frac{d}{d\lambda}\right)^n \left(\frac{1}{\lambda} B_\lambda\right) \right\| &= \left\| \Sigma_{\nu=0}^{n} \binom{n}{\nu} \left(\frac{1}{\lambda}\right)^{(\nu)} \left(\frac{d}{d\lambda}\right)^{n-\nu} B_\lambda \right\| \\ &\leq \Sigma_{\nu=0}^{n} \binom{n}{\nu} \nu! \left(\frac{1}{\lambda}\right)^{\nu+1} (n-\nu)! M \left(\frac{1}{\lambda - \omega}\right)^{n-\nu+1} \\ &\leq \Sigma_{\nu=0}^{n} n! M \left(\frac{1}{\lambda - \omega}\right)^{n+2} \\ &= (n+1)! M \left(\frac{1}{\lambda - \omega}\right)^{n+2} .\end{aligned}$$

Since $(\lambda - \omega)^{n+2} \geq (n+2)\ (\lambda - \omega - 1)^{n+1}$ we finally obtain

$$\leq n! M \left(\frac{1}{\lambda - \omega - 1}\right)^{n+1} .$$

This shows the assertion.

In section 7 of part I we constructed C_o-semigroups that are connected with exponentially bounded, non-degenerate integrated semigroups. Here we try to interpret these in the context of Cauchy problems of second order.

Theorem 2.11:

Let $A_i : D(A_i) \to E$ $(i = 0,1)$ be linear densely defined operators and $\mathcal{A} : D(\mathcal{A}) \to E \times E$ the operator matrix from remark 2.5.

Equivalent are:

(i) $\mathcal{A}$ is an L-generator.

(ii) $\rho(\mathcal{A}) \neq \emptyset$ and $D(\mathcal{A}) = D(A_o) \times D(A_1)$ can be equipped with a norm $\| \ \|_R$ such that:

1) $\|x\| + \|y\| \leq \|(x,y)\|_R \leq c(\|x\|_A + \|y\|_A)$ for $x \in D(A_o)$, $y \in D(A_1)$.

2) The part of $\mathcal{A}$ in $\overline{D(\mathcal{A})}^R$ is generator of a C_o-semigroup on $(\overline{D(\mathcal{A})}^R, \| \ \|_R)$.

The proof is an immediate consequence of part I, theorem 7.2.

This means the following:

The wellposed Cauchy problem of second order can (under the conditions of the theorem) be reformulated to the first order system

$$\frac{d}{dt}\begin{pmatrix} u_t \\ u_t' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ A_o & A_1 \end{pmatrix} \begin{pmatrix} u_t \\ u_t' \end{pmatrix}$$

$$\begin{pmatrix} u_o \\ u_o' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

where the underlying Banach space has to be chosen suitably.

The norm $\| \ \|_R$ can be described more specificly for the "incomplete" Cauchy problem of second order (**):

In theorem 7.1 the norm $\| \ \|_R$ was defined by

$\|x\|_R := \sup_{t\geq 0} \|e^{-\omega t} S_t' x\|$, (where $\|S_t'\| \leq Me^{\omega t}$).

Here

$$S_t' = \begin{pmatrix} \mathrm{Cos}(t) & \mathrm{Sin}(t) \\ \mathrm{Cos}'(t) & \mathrm{Cos}(t) \end{pmatrix}.$$

One verifies easily that $\| \ \|_R$ is equivalent with the norm $\| \ \|_{R'}$, where

$$\|(x,y)\|_{R'} := \|x\|_w + \|y\| \quad \text{and}$$

$$\|x\|_w := \|x\| + \sup_{t\geq 0} \|e^{-\omega t}\mathrm{Cos}'(t)x\| .$$

In other words: The norm can be represented by a product norm on the product $D(A)\times E$, where $A:D(A) \to E$ is the ge-

nerator of the cosine function and $\overline{D(A)\times E}^R = \overline{D(A)}^W \times E$. Very often $A = B^2$ where B is the generator of a C_o-group. The connection between the cosine function and the group is then given by $\mathrm{Cos}(t) = \frac{1}{2}(e^{tB} + e^{-tB})$.

From this $\mathrm{Cos}'(t) = \frac{1}{2}B(e^{tB} + e^{-tB})$ on $D(B)$.

The norm $\| \|_W$ therefore fulfills

$$\|x\|_W \leq c\|x\|_B \quad \text{for} \quad x \in D(A).$$

This implies $D(B) \times E \subset \overline{D(A)}^W \times E$.

Since, on the other hand, $(D(B) \times E, \| \|_B + \| \|_E)$ itself is a reduction space we can describe the connection like this:

$(\overline{D(A)}^W \times E, \| \|_R)$ is (if not a generalization then at least) a "generalizing replacement" of the standard reduction $(D(B) \times E, \| \|_B + \| \|_E)$.

Perturbations:

In theorem 2.9 we formulated sufficient conditions for wellposedness of the Cauchy problem (*) and for an exponentially bounded growth function.

The trouble with these "Hille-Yosida-type" conditions is that in applications they are hard to verify.

We therefore pick up the solvability problem again and try to give (partial) answers in form of perturbation theorems.

The complete Cauchy problem (*) can be approached from two sides:

1) By perturbing the incomplete Cauchy problem (**) by a term $Bu'(t)$ or, more precisely,

by perturbing the L-generator $\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}$ by an operator $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$.

2) by perturbing the Cauchy problem

$$\begin{aligned} u''(t) &= Au'(t) \\ u(0) &= x \\ u'(0) &= y \end{aligned}$$

by a term $Bu(t)$ i.e. perturbing the L-generator $\begin{pmatrix} 0 & 1 \\ 0 & A \end{pmatrix}$ by $\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$.

In both cases the unperturbed Cauchy problem is solved for a large class of operators.

The following theorem shows what one obtains by perturbing $\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$.

Theorem 2.12:

Let A be the generator of a cosine function and $B \in L(E)$.

Then $\begin{pmatrix} 0 & 1 \\ A & B \end{pmatrix}$ is an L-generator.

Proof:

$\mathcal{A} := \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} : D(A) \times E \to E \times E$ is the L-generator of an

integrated semigroup $(S_t)_{t\geq 0}$ (theorem 2.1).

The bounded operator $B := \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ fulfills

$B(E \times E) \subset D(A)$ and $AB = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in L(E \times E)$.

Therefore $S_t'B = (S_tA + I)B = S_tAB + B$ and $(S_t'B)_{t\geq 0}$ is exponentially bounded on $E \times E$. Theorem 6.4 yields the assertion.

Before addressing the second approach "perturbing $\begin{pmatrix} 0 & 1 \\ 0 & A \end{pmatrix}$ by $\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$" we analyse the unperturbed Cauchy problem.

Theorem 2.13:

Let $A:D(A) \to E$ be linear (not necessarily densely defined).

Equivalent are:

(i) $\mathcal{A} := \begin{pmatrix} 0 & 1 \\ 0 & A \end{pmatrix}$ is an L-generator.

(ii) A is an L-generator.

Proof:

(i) => (ii): The integrated semigroup $(T_t)_{t\geq 0}$ that belongs to $\mathcal{A}$ is of the form $T_t = \begin{pmatrix} U_t & V_t \\ P_t & Q_t \end{pmatrix}$ for operators $U_t, V_t, P_t, Q_t \in L(E)$. For large enough λ the operator $\lambda - \mathcal{A}$ is invertible. Evaluating the matrix equations $(\lambda - \mathcal{A})(\lambda - \mathcal{A})^{-1} = I$, $(\lambda - \mathcal{A})^{-1}(\lambda - \mathcal{A}) = I_{|E \times D(A)}$

one obtains that $\lambda - A$ is invertible, too, and

$$(\lambda - A)^{-1} = \begin{pmatrix} \frac{1}{\lambda} & \frac{1}{\lambda}(\lambda - A)^{-1} \\ 0 & (\lambda - A)^{-1} \end{pmatrix} .$$

The equation $(\lambda - A)^{-1} = \lambda\int_0^\infty e^{-\lambda t}T_t \, dt$ implies in particular $(\lambda - A)^{-1} = \lambda\int_0^\infty e^{-\lambda t}Q_t \, dt$ i.e. $(Q_t)_{t\geq 0}$ is an integrated semigroup and A is its L-generator.

(ii) => (i): Let $(S_t)_{t\geq 0}$ be the integrated semigroup for A. Consider the family $(T_t)_{t\geq 0}$, $T_t := \begin{pmatrix} t & \int_0^t S_s \, ds \\ 0 & S_t \end{pmatrix}$.

$(T_t)_{t\geq 0}$ is exponentially bounded and a short calculation shows that

$$\lambda\int_0^\infty e^{-\lambda t}T_t \, dt = \begin{pmatrix} \frac{1}{\lambda} & \frac{1}{\lambda}(\lambda - A)^{-1} \\ 0 & (\lambda - A)^{-1} \end{pmatrix} .$$

One verifies that the right hand matrix is just the inverse of $\lambda - A$. Therefore A is the L-generator of the integrated semigroup $(T_t)_{t\geq 0}$.

Theorem 2.14:

Let $A:D(A) \to E$ be an L-generator and $B \in L(E)$.

Then $\begin{pmatrix} 0 & 1 \\ B & A \end{pmatrix} : E \times D(A) \to E \times E$ is an L-generator.

Remark 2.15:

In the theorem it is not assumed that A is densely defined. Therefore it also makes a statement about Cauchy problems of second order with non-dense domain. It would lead to far to go into this in more detail. Examples of

first order Cauchy problems with non-dense domain are given in section 4.

Proof:

By theorem 2.13 $\begin{pmatrix} 0 & 1 \\ 0 & A \end{pmatrix}$ is an L-generator. Proceding like in the proof of theorem 2.12 one deduces the assertion by theorem 6.7.

Section 3. Perturbing the boundary conditions

In applications generators of C_o-semigroups are very often given by some differential operator the "natural" or "maximal" domain of which is restricted by one or several boundary conditions.

On an abstract level the situation can be described like this:

One has operators

$A:\mathcal{D} \to E$ ("maximal operator") and $L:\mathcal{D} \to Y$ ("boundary functional") and the generator is given as $A_{|\ker L}$.

The question is:

For what pairs (A,L) is $A_{|\ker L}$ a generator?

G. Greiner([14]) solved this problem for a large class of these pairs. Thereby he made use of an observation:

> For many operators A at least <u>one</u> boundary functional L_o is known that leads to a generator.

His procedure was this: He started with these "solved" cases and gained other territory by perturbing the functional L_o, i.e. he proved perturbation theorems of the form:

Let $A_{|\ker L}$ be a generator and $\phi: \mathcal{D} \to Y$ a functional. Under certain conditions on A, L and ϕ the restriction $A_{|\ker(L-\phi)}$ is a generator, too.
Another observation was crucial:

> "Normally" the boundary functional L_o has a closed range in the (Banach) space of boundary functions, or: There is no big loss in assuming that L_o is surjective.
> Further the space $\mathcal{D}$ can be equipped with a norm such that $\mathcal{D}$ is a Banach space, continuously embedded in E and
> $A \in L(\mathcal{D}, E)$, $L_o \in L(\mathcal{D}, Y)$.

This observation yielded the framework for a series of theorems about perturbations of boundary functionals. We summarize this in

Definition 3.1:

> Let $\mathcal{D}, E$ and Y be Banach spaces, $\mathcal{D}$ dense and continuously embedded in E.
> Let $A \in L(\mathcal{D}, E)$ and $L \in L(\mathcal{D}, Y)$, L surjective.
> If $A_o := A_{|\ker L}$ generates a C_o-semigroup then (A, L) is called a G-pair on $E \times Y$.

The connection between G-pairs and L-generators is given in the following theorem.

Theorem 3.2:

Let $\mathcal{D}$, E and Y be Banach spaces, $\mathcal{D}$ dense and continuously embedded in E.

Let $A \in L(\mathcal{D},E)$ and $L \in L(\mathcal{D},Y)$.

Equivalent are:

(i) (A,L) is a G-pair.

(ii) The operator

$$C: \mathcal{D} \times \{0\} \to E \times Y: (x,0) \to (Ax,Lx)$$

is an L-generator on the Banach space $E \times Y$ and $E \times \{0\} \subset C^1$.

Proof:

(i) => (ii): We show first: $\rho(A_0) \subset \rho(C)$.

For this purpose we show:

If $\lambda \in \rho(A_0)$ then $\mathcal{D} = \ker(\lambda - A) \oplus \ker L$.

Let $x \in \ker(\lambda - A) \cap \ker L$.

Since $\ker L = D(A_0)$ and $\lambda - A_0$ is injective we obtain $x = 0$.

Let $x \in \mathcal{D}$. Then for some $y \in D(A_0)$ $(\lambda - A)x = (\lambda - A_0)y$. It is easy to see that the splitting $x = (x - y) + y$ does the desired.

The operator $L_{|\ker(\lambda - A)}$ is a bounded operator from the Banach space $(\ker(\lambda - A), \| \|_{\mathcal{D}})$ into $(Y, \| \|_Y)$.

$\mathcal{D} = \ker(\lambda - A) \oplus \ker L$ shows that $L_{|\ker(\lambda - A)}$ is injective and by assumption (and this direct sum) it is also surjective. We denote the inverse by L_λ.

L_λ is a bounded operator from Y into $\mathcal{D}$ hence also from Y into E since $\mathcal{D}$ is continuously embedded in E.

We claim:

The inverse of $\lambda - C$ is given by

$$R_\lambda(y,z) := (\lambda - A_o)^{-1}y - L_\lambda z$$

(It would be more precise to write

$R_\lambda(y,z) := ((\lambda - A_o)^{-1}y - L_\lambda z, 0)$ but for simplicity we dropped the second component, i.e. we identified $E\times\{0\}$ with E as we shall also do in the future.)

This holds because

$$\begin{aligned}
&(\lambda - A)\, R_\lambda\,(y,z) \\
&= ((\lambda - C)[(\lambda - A_o)^{-1}y - L_\lambda z], L[(\lambda - A_o)^{-1}y - L_\lambda z]) \\
&= (y,z) \quad \text{(by the definitions of } L_\lambda \text{ and } D(A_o)).
\end{aligned}$$

Let $x \in \mathcal{D}$. Then for suitable $u \in \ker(\lambda - A)$ and $v \in \ker L$ we have $x = u + v$ and therefore

$$\begin{aligned}
R_\lambda(\lambda - C)\,(x,0) &= R_\lambda((\lambda - A)x, -Lx) \\
&= (\lambda - A_o)^{-1}(\lambda - A)x + L_\lambda Lx \\
&= (\lambda - A_o)^{-1}(\lambda - A)v + L_\lambda Lu \\
&= v + u = x .
\end{aligned}$$

So $\rho(A_o) \subset \rho(C)$.

We further have: The part of C in E is A_o. This can be seen in a short contemplation about the nature of C.

We arrive at the following situation:

$\rho(C) \neq \emptyset$ and $C_{\overline{D(C)}} = A_o$ is generator of a C_o-semigroup.

Applying the lemmas 7.3 and 7.4 we obtain that C is an L-generator.

We still have to show: $E \subset C^1$.

Let $(S_t)_{t\geq 0}$ be the integrated semigroup that belongs to the operator C.

$S_t E \subset C^1 \subset \overline{D(C)} = E$ (the first inclusion holds by prop.2.2(i) and the second by the second diagram in part I, section 3). The integrated semigroup $(S_{t|E})_{t\geq 0}$ has the L-generator C_E which as we mentioned earlier is just A_o. On the other hand the integrated semigroup $(T_t)_{t\geq 0}$, $T_t := \int_o^t e^{sA_o} ds$ also has the L-generator A_o.
Therefore both integrated semigroups coincide. This implies: $(S_t)_{t\geq 0}$ is continuously differentiable on E or $E \subset C^1$.

(ii) =>(i):
$E \subset C^1$ implies (by $\overline{D(C)} = E \subset C^1 \subset \overline{D(C)}$): C^1 is closed and $C^1 = E$. By remark 2.4 (part I) we obtain: $(S'_t)_{t\geq 0}$ is a C_o-semigroup on E. Its generator is $C_E = A_o$.
To see that L is surjective notice that for large enough λ $\lambda - C$ is surjective. For points $(0,z)$ the representation $(0,z) = (\lambda - C)(x,0)$ implies $z = Lx$. This completes the proof.

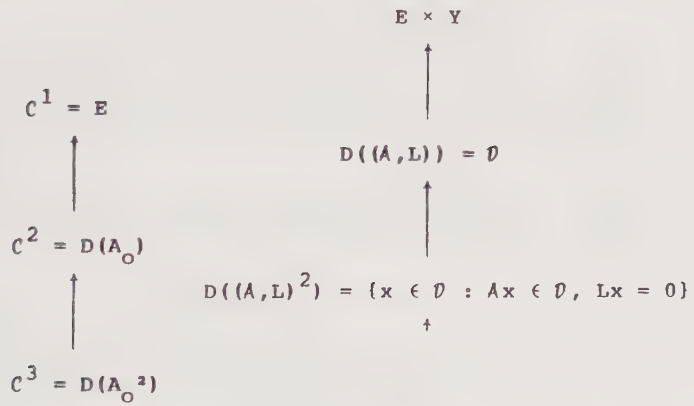
For more transparence we denote from now on the operator C by (A,L).

Remark 3.3:

1) If (A,L) is a G-pair then for any $\alpha \in \mathbb{C}$, $\alpha \neq 0$ $(A,\alpha L)$ is also a G-pair.
The generators $A_{|\ker(\alpha L)}$ are the same for all such α's. But the integrated semigroups depend on α. And for some

α's they have more agreeable properties than for others. Example: $A = \frac{d}{dx}: C^1[0,1] \to C[0,1]$ and $L: C^1[0,1] \to \mathbb{C}$, $Lf := f'(0)$ give a G-pair on $C[0,1] \times \mathbb{C}$ (resp. an L-generator). $(A,-L)$ is an L-generator, too, but with an additional property: $(A,-L)$ is the L-generator of a positive (and increasing) integrated semigroup. This property plays a role when positive perturbations of boundary functionals are considered.

2) We go back to the diagrams of part I, section 3. Specifying the first diagram we arrive at



Here the differentiation spaces are just the spaces $D(A_o^n)$ where A_o is the generator obtained by restricting A. A non-trivial boundary condition $Lx = 0$ (i.e. a boundary condition that really reduces $\mathcal{D}$) is reflected in the diagram as a proper inclusion $C^2 = D(A_o) \subsetneq \mathcal{D}$. The isomorphisms $\lambda - (A,L)$ (denoted by the arrows) imply that $C^n \subsetneq D((A,L)^{n-1})$ for all $n \geq 1$. If a non-trivial boun-

dary condition leads to a generator then A must be unbounded or $\mathcal{D} \subsetneq E = C^1$. We obtain by the same argument as before that $D((A,L)^n) \subsetneq C^n$ for all $n \geq 1$. This shows that in case of non-trivial boundary conditions all inclusions are proper.

We illustrate in two theorems how to prove perturbation theorems for boundary functionals within this framework.

Theorem 3.4 (Greiner [14], theorem 2.1):

Let (A,L) be a G-pair on $E \times Y$.

Suppose there are constants $\gamma, \omega \in \mathbb{R}$ with the property: If $\lambda > \omega$ and $x \in \ker(\lambda - A)$ then

$$\|Lx\|_Y \geq \lambda\gamma\|x\|_E .$$

Then for any $\phi \in L(E,Y)$ $A|_{\ker(L-\phi)}$ generates a C_o-semigroup.

Proof:

In the proof of theorem 3.2 we showed: The resolvent of the L-generator (A,L) is given by

$$R_\lambda(y,z) = (\lambda - A_o)^{-1}y - L_\lambda z$$

The assumption means: $\|L_\lambda\|_{L(Y,E)} \leq \gamma/\lambda$ for $\lambda > \omega$.

Let M,ω' be constants such that $\|e^{tA}\| \leq Me^{\omega' t}$ for $t \geq 0$.

Then

$$\|R_\lambda(y,z)\|_{E\times Y} = \|(\lambda - A_o)^{-1}y - L_\lambda z\|_E$$

$$\leq \frac{M}{(\lambda-\omega')} \|y\|_E + \frac{\gamma}{\lambda} \|z\|_Y$$

$$\leq \frac{M''}{(\lambda - \omega'')} \|(y,z)\|_{E\times Y}$$

and from this

$$\|R_\lambda^n (y,z)\| = \|R_\lambda^{n-1} R_\lambda (y,z)\|_{E\times Y}$$

$$= \| (\lambda - A_o)^{-n+1} R_\lambda (y,z)\|_E ,$$

since $R_\lambda\ E \times Y \subset E$

$$\leq M/(\lambda - \omega')^{n-1} \quad M''/(\lambda - \omega'') \|(y,z)\|_{E\times Y}$$

$$\leq MM''/(\lambda - \omega'')^n \|(y,z)\|_{E\times Y}.$$

By theorem 4.3 of part I this implies: (A,L) is L-generator of a locally Lipschitz-continuous integrated semigroup.

By $B(y,z) := (0,-\phi y)$ a bounded operator is defined on $E \times Y$.

By theorem 6.5 we deduce: $(A,L) + B = (A,L - \phi)$ is the L-generator of a locally Lipschitz-continuous integrated semigroup.

And the additional assertion of theorem 6.5 implies:
$E \subset C^1 \subset C_B^1$.

Theorem 3.2 completes the proof.

Theorem 3.5 (Greiner [14], theorem 2.3):

Let (A,L) be a G-pair on $E \times Y$,

$\phi \in L(E,Y)$ and $c \geq 0$ such that

$\|\phi A x\|_Y \leq c\|x\|_E$ for all $x \in D$

Then $A_{|\ker(L-\phi)}$ generates a C_o-semigroup.

Proof:

By theorem 3.2 (A,L) is an L-generator.

Consider again the bounded operator B, $B(y,z) := (0,\phi y)$.

By assumption $B \circ (A,L)$ is bounded.

Theorem 6.7 implies that $(A,L) + B = (A,L - \phi)$ is an L-generator.

And the additional assertion of theorem 6.7 yields: $E = C^1 \subset C^1_B$. Using again theorem 3.2 we obtain the assertion.

Section 4. Cauchy problems with non-dense domain

In this section we consider Cauchy problems

$$(*) \qquad \begin{aligned} \frac{d}{dt}u(t) &= Au(t) + f(t) \\ u(0) &= x \end{aligned}$$

where $A:D(A) \to E$ is a linear but not necessarily densely defined operator, $x \in D(A)$ and $f:[0,1] \to E$.
A solution (*) is a C^1-function $u:[0,1] \to E$ such that $u(t) \in D(A)$ for $0 \le t \le 1$ and (*) holds.

Theorem 4.1 (uniqueness and mild solutions):

Let A be L-generator of the integrated semigroup $(S_t)_{t\ge 0}$, $f:[0,1] \to E$ continuous and $x \in D(A)$.
If the Cauchy problem (*) is solvable the solution is given by

$$u(t) = S_tAx + x + \frac{d}{dt}\left(\int_0^t S_{t-s}f(s)\,ds\right).$$

Proof:

Lemma 6.6 (part I) and the fact that $AS_t = S_tA$ on $D(A)$ imply $S_t'x = S_tAx + x$ for $x \in D(A)$. This will be needed in the following.

Let u be a solution of (*).

Then the function $s \to S_{t-s}u(s)$ is continuously differentiable and

$$\begin{aligned} \frac{d}{ds}(S_{t-s}u(s)) &= -S'_{t-s}u(s) + S_{t-s}u'(s) \\ &= -S_{t-s}Au(s) - u(s) + S_{t-s}Au(s) + S_{t-s}f(s) \\ &= S_{t-s}f(s) - u(s). \end{aligned}$$

Integration from 0 to t yields (recall that $S_o = 0$ and $u(0) = x$):

$$\int_o^t u(s)\,ds - S_t x = \int_o^t S_{t-s} f(s)\,ds.$$

The left hand side is differentiable in t, therefore also the right hand side and one obtains

$$u(t) - S_t' x = \frac{d}{dt}\left(\int_o^t S_{t-s} f(s)\,ds\right).$$

Together with $S_t' x = S_t Ax + x$ this implies the assertion.

Theorem 4.2 (Arendt [5], theorem 8.1):

Let $A:D(A) \to E$ be an L-generator, $f:[0,1] \to E$ a C^2-function, $x \in D(A)$ and $Ax + f(0) \in C^1$.

Then (*) is uniquely solvable.

The assumptions in [5], theorem 8.1 are slightly stronger but can be weakened in the above sense without changing the proof.

Theorem 4.3 (Da Prato - Sinestrari [10]):

Let $A:D(A) \to E$ be a linear operator, $f:[0,1] \to E$ and $x \in D(A)$ such that

a) there are $M, \omega \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ and $\lambda > \omega$

$$\|(\lambda - A)^{-n}\| \leq M/(\lambda - \omega)^n .$$

b) $f(t) = f(0) + \int_o^t g(s)\,ds$

for some Bochner-integrable function g.

c) $Ax + f(0) \in \overline{D(A)}$.

Then the Cauchy problem (*) is uniquely solvable.

Proof:

Assumption a) means by theorem 3.3: A is L-generator of a locally Lipschitz-continuous integrated semigroup $(S_t)_{t\geq 0}$.

We show: The solution is given by

$$u(t) = S_t(Ax + f(0)) + x + \int_0^t S_{t-s}g(s)\,ds.$$

We split u and the assertion in two parts:

1) $v(t) := S_t(Ax + f(0)) + x$ solves the Cauchy problem

$$v'(t) = Av(t) + f(0),$$
$$v(0) = x.$$

2) $w(t) := \int_0^t S_{t-s}g(s)\,ds$ solves the Cauchy problem

$$w'(t) = Aw(t) + \int_0^t g(s)\,ds,$$
$$w(0) = 0.$$

1): By remark 5.4 of part I $C^1 = \overline{D(A)}$.

This implies with assumption c) that v is continuously differentiable and

$$v'(t) = S_t'(Ax + f(0)) = AS_t(Ax + f(0)) + Ax + f(0)$$
$$= Av(t) + f(0).$$

Obviously $v(0) = x$. Therefore 1) holds.

2):

We show first: For every t the function $s \to S_{t-s}g(s)$ is Bochner-integrable.

Let (g_n) be a sequence of step functions such that $g_n \to g$ a.e. and $\int_0^t \|g_n(s) - g(s)\|\,ds \to 0$.

All functions g_n are of the form $\sum_{i=1}^{N} \chi_{[t_i, t_{i+1})}x_i$.

Consider, for fixed n, the refining partition for $0 \leq t_1 \leq \ldots \leq t_k \leq t$ and $0 < \frac{1}{n} < \frac{2}{n} < \ldots < t$. Denote it by $0 \leq s_1 \leq \ldots \leq s_m \leq t$.

Let L be the Lipschitz-constant of $(S_t)_{0\le t\le 1}$.

The functions $v_n := \Sigma_{i=1}^{m} \chi_{[s_i, s_{i+1})} S_{t-s_i} x_i$ yield the approximating sequence of step functions for the function $s \to S_{t-s}g(s)$. To see this let $0 \le r \le t$.

For a suitable $s_k \le r \le s_{k+1}$.

Therefore

$$\begin{aligned} & \| v_n(r) - S_{t-r}g(r) \| \\ = \; & \| S_{t-s_k}(x_k - g(r)) + (S_{t-s_k} - S_{t-r})g(r) \| \\ = \; & \| S_{t-s_k}(g_n(r) - g(r)) + (S_{t-s_k} - S_{t-r})g(r) \| \\ \le \; & L \, \| g_n(r) - g(r) \| + L|r - s_k| \, \| g(r) \| \\ \le \; & L \, \| g_n(r) - g(r) \| + L/n \, \| g(r) \| \to 0 \end{aligned}$$

a.e. in r as $n \to \infty$.

Further:

$$\begin{aligned} & \int_0^t \| v_n(r) - S_{t-r}g(r) \| \, dr \\ \le \; & L \int_0^t \| g_n(r) - g(r) \| \, dr + L/n \int_0^t \| g(r) \| \, dr \to 0. \end{aligned}$$

This shows that $s \to S_{t-s}g(s)$ is Bochner-integrable and that w in 2) is a legitimate expression.

We show 2) in two steps:

a) For the case: g is a step function.

b) For the general situation (by an approximation).

a): Let $g = \Sigma_{i=1}^{N} \chi_{[t_i, t_{i+1})} x_i$.

Then

$$\begin{aligned} w(t) := & \int_0^t S_{t-s}g(s) \, ds \\ = & \Sigma_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} S_{t-s}x_i \, ds + \int_{t_k}^{t} S_{t-s}x_k \, ds \end{aligned}$$

(t_k denotes the greatest $t_i < t$)

$$= \Sigma_{i=1}^{k-1} \int_{t-t_{i+1}}^{t-t_i} S_r x_i \, dr + \int_0^{t-t_k} S_r x_k \, dr.$$

From this we obtain

$$\frac{d}{dt}w(t) = \Sigma_{i=1}^{k-1} (S_{t-t_i} - S_{t-t_{i+1}})x_i + S_{t-t_k}x_k$$

and by lemma 6.6 of part I

$$= \Sigma_{i=1}^{k-1} (A \int_o^{t-t_i} S_r x_i \, dr + (t - t_i)x_i)$$
$$- \Sigma_{i=1}^{k-1} (A \int_o^{t-t_{i+1}} S_r x_i \, dr + (t - t_{i+1})x_i)$$
$$+ A \int_o^{t-t_k} S_r x_k \, dr + (t - t_k)x_k$$
$$= Aw(t) + \Sigma_{i=1}^{k-1} (t_{i+1} - t_i)x_i + (t - t_k)x_k$$
$$= Aw(t) + \int_o^t g(s) \, ds.$$

b) (approximation):

Let g be Bochner-integrable and (g_n) a sequence of step functions converging to g in the sense described above. We show: If for every g_n a function w_n as in a) is defined then the derivatives w_n' form a Cauchy sequence.

Let $m,n \in \mathbb{N}$.

We write g_m in the form $g_m(t) = \Sigma_{i=1}^{N} \chi_{[t_i, t_{i+1})} x_i$ and g_n in the form $g_n(t) = \Sigma_{i=1}^{N} \chi_{[t_i, t_{i+1})} y_i$ i.e. we drop the indices m and n and go over to the refining partition $0 \le t_1 \le \ldots \le t_N \le 1$.

For fixed t let t_k be the greatest partition point below t. According to what we just showed

$$w_m'(t) - w_n'(t)$$
$$= \Sigma_{i=1}^{k-1} (S_{t-t_i} - S_{t-t_{i+1}})x_i + S_{t-t_k}x_k$$
$$- \Sigma_{i=1}^{k-1} (S_{t-t_i} - S_{t-t_{i+1}})y_i - S_{t-t_k}y_k$$
$$= \Sigma_{i=1}^{k-1} (S_{t-t_i} - S_{t-t_{i+1}})(x_i - y_i) + S_{t-t_k}(x_k - y_k).$$

Froms this

$$\|w_m'(t) - w_n'(t)\|$$
$$\le \Sigma_{i=1}^{k-1} \|S_{t-t_i} - S_{t-t_{i+1}}\| \|x_i - y_i\|$$
$$+ \|S_{t-t_k}\| \|x_k - y_k\|$$
$$\le L \int_o^t \|g_m(s) - g_n(s)\| \, ds \to 0 \quad \text{as} \quad m,n \to \infty .$$

We therefore have

$w_n' \to h$ (uniformly in t) for some continuous funktion $h:[0,1] \to E$.

Since w_n converges to w (which holds because $\|w_n(t) - w(t)\| = \|\int_0^t S_{t-s}(g_n(s) - g(s))\, ds\|$) we finally have: w is continuously differentiable in t and $w' = h$.

All functions fulfill the assertion 1):

$$w_n'(t) = Aw_n(t) + \int_0^t g_n(s)\, ds .$$

With $n \to \infty$ and the closedness of A this implies

$$w'(t) = Aw(t) + \int_0^t g(s)\, ds$$

and the assertion.

(The uniqueness follows from theorem 4.1.)

We conclude this section by

Example 4.4 (Cauchy problems with inhomogeneous boundary conditions:

Let $A:\mathcal{D} \to E$ be a densely defined operator in a Banach space and $L:\mathcal{D} \to Y$ a boundary functional (see section 3).

We consider Cauchy problems

$$\begin{aligned} u'(t) &= Au(t) + g(t) \\ u(0) &= x \\ Lu(t) &= h(t), \end{aligned} \tag{**}$$

where $g:[0,1] \to E$ and $h:[0,1] \to Y$.

A solution is a C^1-function $u:[0,1] \to E$ such that $u(t) \in \mathcal{D}$ and (**) holds.

We combine the preceding results with the results in section 3 to get two existence theorems for (**).

Theorem 4.5:

Let E and Y be Banach spaces, $A:\mathcal{D} \to E$ and $L:\mathcal{D} \to Y$ be such that (A,L) is a G-pair.

Let $g:[0,1] \to E$ and $h:[0,1] \to Y$ be C^2-functions and $x \in \mathcal{D}$ such that $Lx = h(0)$.

Then (**) is uniquely solvable.

Proof:

Write (**) in the form

$$v'(t) = (A,-L)v(t) + (g(t),h(t))$$
$$v(0) = (x,0).$$

By theorem 3.2 $(A,-L)$ is an L-generator on $E \times Y$ with $E\times\{0\} \subset C^1$. Together with the assumptions $x \in \mathcal{D}$ and $Lx = h(0)$ this implies $(A,-L)(x,0) + (g(0),h(0)) \in E\times\{0\} \subset C^1$. Now the assertion follows by theorem 4.2.

Theorem 4.6:

Let E and Y be Banach spaces and (A,L) a G-pair such that

$\|Lx\|_Y \geq \gamma(\lambda - \omega)\|x\|_E$ for $\lambda > \omega$ and $x \in \ker(\lambda - A)$

(γ and ω are suitable constants).

Let $g:[0,1] \to E$ be of the form

$g(t) = g(0) + \int_0^t \bar{g}(s)\,ds$ for some Bochner-integrable function $\bar{g}$ and let the same hold for $h:[0,1] \to Y$.

If $x \in \mathcal{D}$ such that $Lx = h(0)$ then (**) is uniquely solvable.

Proof:

The proof of theorem 3.2 shows:

The estimate for L implies that $(A,-L)$ fulfills the Hille-Yoshida-estimates required in theorem 4.3a). Since all other assumptions are fulfilled, too, the assertion follows by theorem 4.3.

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